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A nodec regular analytic topology

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ABSTRACT

A topological space X is said to be *maximal* if its topology is maximal among all T_1 topologies over X without isolated points. It is known that a space is maximal if, and only if, it is extremely disconnected, nodec and every open set is irresolvable. We present some results about the complexity of those properties on countable spaces. A countable topological space X is analytic if its topology is an analytic subset of $\mathcal{P}(X)$ identified with the Cantor cube $\{0,1\}^X$. No extremely disconnected space can be analytic and every analytic space is hereditarily resolvable. However, we construct an example of a nodec regular analytic space.

1. Introduction

A space is called *maximal* if its topology is maximal in the collection of T_1 topologies without isolated points. In [3] it was shown that a topology is maximal if, and only if, it is extremely disconnected (i.e. the closure of every open set is open), nodec (i.e. every nowhere dense set is closed) and every open set is irresolvable (i.e. if U is open and $D \subseteq U$ is dense in U, then $U \setminus D$ is not dense in U). In the same article, van Douwen constructed a countable space which is maximal and regular. In this note we are interested in the complexity of such maximal countable spaces. The complexity is measure in the sense of the projective hierarchy as we explain below.

A subset A of a metric space is called *analytic* if it is a continuous image of a Polish space. Equivalently, if there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ with range A, where $\mathbb{N}^{\mathbb{N}}$ is the space of irrationals [5]. Every Borel subset of a Polish space is analytic. We say that a topology τ over a countable set X is *analytic*, if τ is analytic as a subset of the Cantor cube 2^X (identifying subsets of X with characteristic functions) [9,10].

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Clearly, every first countable topology over X is Borel (in fact, $F_{\sigma\delta}$). The topology of the sequential fan, Arens space and Arkhanglel'skiĩ–Franklin space are also examples of Borel topologies.

It is known that an analytic Hausdorff extremely disconnected space must be discrete [10, p. 521]. Hence there are no Hausdorff analytic maximal spaces. In this note we will look at the status of analytic nodec and irresolvable spaces. First we show that every T_1 space with an analytic topology is resolvable. All known examples of irresolvable spaces [2] and thus maximal spaces are constructed using the axiom of choice. Our result gives an indication that this is somehow inevitable.

We will show that there is an analytic regular nodec space without isolated points. Our example was motivated by some results in [8]. In a nodec space without isolated points every discrete set is nowhere dense and therefore closed. So a dense in itself nodec space is not discretely generated (recall that a space X is *discretely generated* [4], if for every $A \subseteq X$ and every $x \in \overline{A}$, there is a discrete subset $D \subseteq A$ such that $x \in \overline{D}$). Since the topology of our example is analytic, then $C_p(\mathbb{N}^{\mathbb{N}})$ has a copy of it (see [9, Proposition 6.1]). This in particular says that $C_p(\mathbb{N}^{\mathbb{N}})$ is not discretely generated. We recall that $C_p(X)$ is discretely generated when X is a σ-compact space [1].

2. Resolvability

In this section we show that every analytic space is resolvable.

Theorem 2.1. *Let X be a T_1 countable space without isolated points and with an analytic topology. Then X is resolvable. Moreover, X is $\aleph_0$-resolvable, i.e. there is a countable family of disjoint dense sets. In particular, every nonempty open subset of X is resolvable.*

Proof. Let $\{x_j\}_j$ be an enumeration of X. Consider the ideal $\mathcal{I}_j$ on $\mathbb{N}\setminus\{j\}$ consisting of all $A \subseteq \mathbb{N}\setminus\{j\}$ such that $x_j \notin \overline{\{x_i\colon i \in A\}}$. Since X is T_1 and has no isolated points, then $\mathcal{I}_j$ is a nonprincipal ideal containing every finite set. Moreover, as the topology of X is analytic, then $\mathcal{I}_j$ is also analytic, in particular, $\mathcal{I}_j$ has the Baire property. By a theorem of Jalali-Naini, Talagrand (see, for instance, [7, p. 33]) for each $j \in \mathbb{N}$, there is a pairwise disjoint family $\{A_i^j\}_{i\geqslant 0}$ of finite subsets of $\mathbb{N}\setminus\{j\}$ such that $X\setminus\{x_j\} = \bigcup_i\{x_n\colon n \in A_i^j\}$ and

$$x_j \in \overline{\bigcup_{i\in B}\{x_n\colon n \in A_i^j\}} \quad \text{for every } B \subseteq \mathbb{N} \text{ infinite.}$$

To simplify the notation, we will identify a subset $A \subseteq \mathbb{N}$ with $\{x_i\colon i \in A\}$. By a standard recursive diagonalization procedure, there are two sequences $(n_j)_j$ and $(m_j)_j$ such that

(i) $(n_j)_j$ is increasing,
(ii) for each k, $m_j = k$ for infinitely many j and
(iii) the sets $A_{n_j}^{m_j}$, for $j \in \mathbb{N}$, are pairwise disjoint.

Then $\bigcup_j A_{n_j}^{m_j}$ is dense. Finally, using (ii), it follows that there are two dense disjoint subsets of X. This construction can easily be modified to build a countable collection of disjoint sets. The last claim follows from the first, since every subspace of an analytic space also has an analytic topology. □

Corollary 2.2. *Every countable subspace of $C_p(Z)$ is resolvable, where Z is a polish space.*

Proof. Every countable subspace of $C_p(Z)$ is analytic [9, Proposition 6.1]. □

Corollary 2.3. *There are no submaximal analytic topologies (i.e. a dense in itself T_1 space where dense sets are open).*