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# Extremal function pairs in asymmetric normed linear spaces ☆

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ABSTRACT

In [4] Kemajou et al. constructed the injective hull in the category of $T_0$-quasi-metric spaces with nonexpansive maps that they called $q$-hyperconvex hull. In this paper, we study properties of functions pairs of the $q$-hyperconvex hull of asymmetric normed linear spaces. We show, for instance, that any point in the $q$-hyperconvex hull of an asymmetric normed linear space is convex.



## 1. Introduction

In [4] Kemajou et al. studied the concept of hyperconvexity that is appropriate to the category of $T_0$-quasi-metric spaces and nonexpansive maps. They provided an explicit construction of the corresponding hull (called $q$-hyperconvex hull, or Isbell-convex hull) of a $T_0$-quasi-metric space. In this paper we study properties of functions pairs in the $q$-hyperconvex hull of an asymmetric normed linear space. We point out that every point in the $q$-hyperconvex hull of an asymmetric normed linear space is convex and we show that the translation of any point in the injective hull of an asymmetric normed linear space is extremal. We hope that the results of this paper will be applied for our further investigations of Banach properties of the $q$-hyperconvex hull of an asymmetric normed linear space.

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## 2. Preliminaries

For the convenience of the reader and in order to fix our terminology we recall the following concepts.

**Definition 1.** Let $X$ be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then $d$ is called a *quasi-pseudometric* on $X$ if

(a) $d(x, x) = 0$ whenever $x \in X$,
(b) $d(x, z) \leqslant d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We shall say that $d$ is a $T_0$-*quasi-metric* provided that $d$ also satisfies the following condition: For each $x, y \in X$,

$$d(x, y) = 0 = d(y, x) \text{ implies that } x = y.$$

**Remark 1.** In some cases we need to replace $[0, \infty)$ by $[0, \infty]$ (where for a $d$ attaining the value $\infty$ the triangle inequality is interpreted in the obvious way). In such a case we shall speak of an *extended quasi-pseudometric*.

**Remark 2.** Let $d$ be a quasi-pseudometric on a set $X$, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric of* $d$. As usual, a quasi-pseudometric $d$ on $X$ such that $d = d^{-1}$ is called a *pseudometric*. Note that for any ($T_0$-)quasi-pseudometric $d$, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

For any $a, b \in \mathbb{R}$, we shall set $a \dot{-} b = \max\{a - b, 0\}$.

The following definition can be found in [2] (compare [3]).

**Definition 2.** Let $X$ be a linear space over $\mathbb{R}$ and let $\|.| : X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then $\|.|$ is called an asymmetric norm on $X$ if

(AN1) $\|x| = \|-x| = 0 \Longrightarrow x = 0$ for all $x \in X$,
(AN2) $\|\alpha x| = \alpha\|x|$ for all $x \in X$ and $\alpha \geqslant 0$,
(AN3) $\|x + y| \leqslant \|x| + \|y|$ for all $x, y \in X$.

The pair $(X, \|.|)$ is called an asymmetric normed linear space.

**Remark 3.** Let $\|.|$ be an asymmetric norm on a linear space $X$ over $\mathbb{R}$, then $|.\| : X \to [0, \infty)$ defined by $|x\| = \|-x|$ whenever $x \in X$ is also an asymmetric norm, called the *conjugate asymmetric norm* of $\|.|$. As usual, an asymmetric norm $\|.|$ on $X$ such that $\|.| = |.\|$ is called a norm. Furthermore, for any asymmetric norm $\|.|$, $\|.\| = \max\{\|.|, |.\|\}$ is a norm and $(X, \|.\|)$ is a normed linear space. The asymmetric norm induces, in a natural way, a quasi-metric $d_{\|.|}$ on $X$ defined by $d_{\|.|}(x, y) = \|x - y|$ for all $x, y \in X$.

The following example is well-known, but important.

**Example 1.** ([2, Example 1.1.3]) Let the set $\mathbb{R}$ of real numbers be equipped with the asymmetric norm $v(x) = x^+ = \max\{x, 0\}$. Then, for $x \in \mathbb{R}$, $v^{-1}(x) = x^- = \max\{-x, 0\}$ and $v^s = |x|$. The topology $\tau(v)$ generated by $v$ is called the *upper topology* of $\mathbb{R}$; while the topology $\tau(v^{-1})$ generated by $v^{-1}$ is called the *lower topology* of $\mathbb{R}$.