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Coloring link diagrams and Conway-type polynomial of braids

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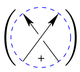
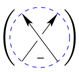
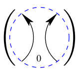
ABSTRACT

In this paper we define and present a simple combinatorial formula for a 3-variable Laurent polynomial invariant $I(a,z,t)$ of conjugacy classes in Artin braid group $\mathbf{B}_m$. We show that the Laurent polynomial $I(a,z,t)$ satisfies the Conway skein relation and the coefficients of the 1-variable polynomial $t^{-k}I(a,z,t)|_{a=1,t=0}$ are Vassiliev invariants of braids.

1. Introduction

In this work we consider link invariants arising from the Alexander–Conway and HOMFLY-PT polynomials. The HOMFLY-PT polynomial $P(L)$ is an invariant of an oriented link L (see for example [10,18,23]). It is a Laurent polynomial in two variables a and z, which satisfies the following skein relation:

aP  $- a^{-1}P$  $= zP$ . (1)

The HOMFLY-PT polynomial is normalized in the following way. If O_r is the r-component unlink, then $P(O_r) = (\frac{a-a^{-1}}{z})^{r-1}$. The Conway polynomial ∇ may be defined as $\nabla(L) := P(L)|_{a=1}$. This polynomial is a renormalized version of the Alexander polynomial (see for example [9,17]). All coefficients of ∇ are finite type or Vassiliev invariants.

Recently, invariants of conjugacy classes of braids received a considerable attention, since in some cases they define quasi-morphisms on braid groups and induce quasi-morphisms on certain groups of diffeomorphisms of smooth manifolds, see for example [3,6–8,11,12,14,15,19,20].

In this paper we present a certain combinatorial construction of a 3-variable Laurent polynomial invariant $I(a,z,t)$ of conjugacy classes in Artin braid group $\mathbf{B}_m$. We show that the polynomial $I(a,z,t)$ satisfies the Conway skein relation and the coefficients of the polynomial $t^{-k}I(a,z,t)|_{a=1,t=0}$ are finite type invariants

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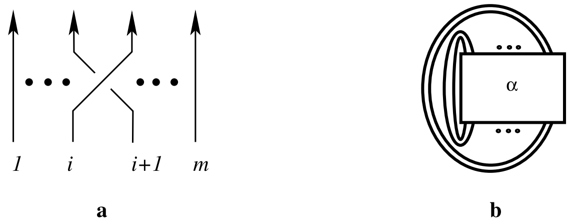

Fig. 1. Artin generator σ_i and a closure of a braid α.

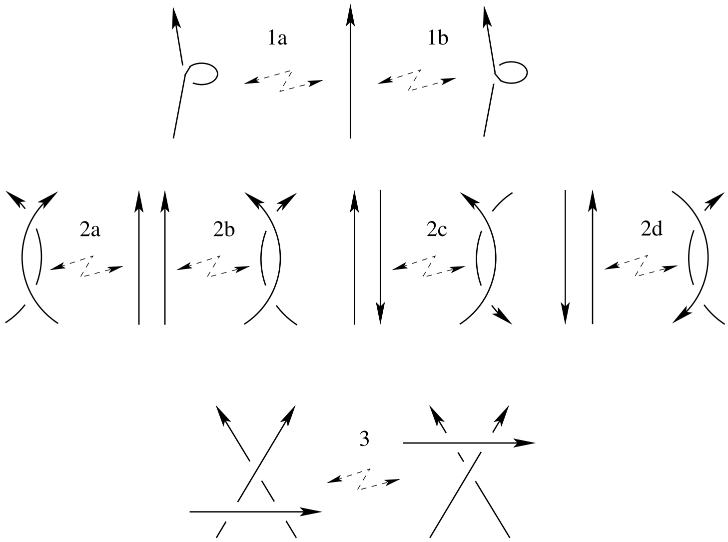

Fig. 2. $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, $\Omega 2b$, $\Omega 2c$, $\Omega 2d$ and $\Omega 3$ Reidemeister moves.

of braids for every $k \geqslant 2$. We modify the polynomial $t^{-2}I(a,z,t)|_{a=1,t=0}$, so that the resulting polynomial is a polynomial invariant of links. In addition, we show that this polynomial equals to $zP'_a|_{a=1}$, where $P'_a|_{a=1}$ is the partial derivative of the HOMFLY-PT polynomial, w.r.t. the variable a, evaluated at $a = 1$. Another interpretation of the later polynomial was recently given by the author in [4,5].

1.1. Construction of the polynomial $I(a,z,t)$

Recall that the Artin braid group $\mathbf{B}_m$ on m strings has the following presentation:

$$\mathbf{B}_m = \langle \sigma_1, \dots, \sigma_{m-1} | \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geqslant 2; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle, \tag{2}$$

where each generator σ_i is shown in Fig. 1a. Let $\alpha \in \mathbf{B}_m$. We take any representative of α and connect it opposite ends by simple nonintersecting curves as shown in Fig. 1b and obtain the oriented link diagram D. We impose an equivalence relation on the set such diagrams as follows. Two such diagrams are equivalent if one can pass from one to another by a finite sequence of $\Omega 2a$, $\Omega 2b$ and $\Omega 3$ Reidemeister moves shown in Fig. 2. It follows directly from the presentation (2) of $\mathbf{B}_m$ that the equivalence class of such diagrams depends on α and does not depend on the representative of α, see for example [16]. It is called the *closed braid* and is denoted by $\widehat{\alpha}$. It is straightforward to show that there is a one-to-one correspondence between the conjugacy classes in the braid groups $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \dots$ and closed braids, see for example [16].

Now we are ready to describe our construction of the polynomial $I(a,z,t)$. We fix a natural number $k \geqslant 2$. Let D be a diagram of an oriented link L. We remove from D a small neighborhood of each crossing, see Fig. 3. The remaining arcs we will color by numbers from $\{1, \dots, k\}$ according to the following rule: the