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Eulerian paths and a problem concerning n-arc connected spaces

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ABSTRACT

In this paper we give, in response to a question of Espinoza, Gartside and Mamatelashvili, an example of an n-arc connected (metric) continuum which is not $(n+1)$-arc connected for every $n \geqslant 7$.

1. Introduction

A topological space X is called n-arc connected (n-ac) if for any points $x_1, \dots, x_n \in X$, there is an arc γ in X such that $x_1, \dots, x_n$ are all in γ [2]. Note that a space is 2-arc connected if and only it is arcwise connected.

If every n-points of a space X lie on an arc which goes through them in order, X will be called n-strong arc connected (n-sac) [3].

A (topological) graph is a connected space obtained by taking a finite nonempty family $\mathcal{F}$ of disjoint arcs (i.e., homeomorphic copies of the unit interval), and then identifying some of the endpoints.

In [2] it is shown that:

i) There are n-arc connected graphs which are not $(n+1)$-arc connected for every $n \leqslant 6$.
ii) A 7-arc connected space which is a graph must be n-arc connected for every n.

This led the authors of [2] to ask for examples of (regular) continua which are n-ac but not $(n+1)$-ac for $n \geqslant 7$. (A continuum is said to be regular if it has a base all of whose elements have a finite boundary.)

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The aim of this note is to give such examples (for every $n \geqslant 2$).

Let G be a graph (given by a finite family $\mathcal{F}$ of arcs):

i) Every arc in $\mathcal{F}$ (modulo the identifications) is called edge. An endpoint (modulo the identifications) of an edge is called vertex. An edge whose vertices are coincident is called loop.
ii) The degree of a vertex is the number of edges incident to the vertex (with loops counted twice).
iii) A path in G is called Eulerian if it visits every edge of G at most once.

We say also that a graph G is n-Eulerian if for any points $x_1, \dots, x_n \in G$, there is an Eulerian path γ in G such that $x_1, \dots, x_n$ are all in γ.

Clearly every n-ac graph is n-Eulerian, moreover a graph G which has an Eulerian path whose image is G (i.e., it is surjective) is n-Eulerian for every n.

Our solution to the problem of Espinoza, Gartside and Mamatelashvili will rely on the following

Euler Theorem. *A graph has a surjective Eulerian path if and only if it has at most two odd vertices* (*i.e., either all vertices are of even degree, or exactly two vertices are of odd degree*).

The reader is referred to [1] for notations and terminology not explicitly given.

2. The results

A pertinent consequence of the Euler theorem cited above is the following

Proposition 1. *Let G be a graph without loops which has exactly four vertices, all of odd degree. If the number of edges is $n+1$, then G is n-Eulerian but not $(n+1)$-Eulerian.*

Proof. Let us show that G is not $(n+1)$-Eulerian. Take $n+1$ points $x_1, \dots, x_{n+1} \in G$, each one in a different edge of G. An Eulerian path in G containing $x_1, \dots, x_{n+1}$ would be surjective, this is not possible (by Euler theorem). Therefore G is not $(n+1)$-Eulerian.

Now let us take n points $x_1, \dots, x_n \in G$. Without loss of generality, we may assume that none of them is a vertex. Then there is an edge γ of G which does not contain any of $x_1, \dots, x_n$.

Now $H = \overline{G \setminus \gamma}$ is connected. In fact, let v_1, v_2, v_3 and v_4 be the vertices of G and let us suppose that v_1 and v_2 are the endpoints of γ. If v_1 (or v_2) is not in H, then H is clearly connected. Otherwise, if $v_1, v_2 \in H$ and H is disconnected, then one of the vertices v_3 and v_4 is joined only with v_1 and the other one only with v_2. Since v_1 and v_2 have odd degree, it follows that v_3 and v_4 have even degree. A contradiction.

So H is a graph with exactly two vertices of odd degree (because G has no loops). Therefore, by Euler theorem, there is a surjective Eulerian path γ in H. So γ is an Eulerian path in G containing $x_1, \dots, x_n$. Therefore G is n-Eulerian. $\square$

Remark 2. Let us note that for every $n \geqslant 3$ there is a graph without loops with exactly four vertices, all of odd degree, and n edges.

The construction is by induction.

For $n = 3$ let us take four vertices v_0, v_1, v_2 and v_3 and three arcs l_1, l_2 and l_3 in such a way that l_i joins v_i and v_0 for $i = 1, 2, 3$.

For $n = 4$ let us take four vertices v_1, v_2, w_1 and w_2 and four arcs l_1, l_2, m_1 and m_2 in such a way that l_i joins v_i and w_i and m_i joins w_1 and w_2 for $i = 1, 2$.

Now given a graph with n edges we may obtain a graph with $n+2$ edges by fixing two distinct vertices v_i and v_j and adding two edges with endpoints v_i and v_j.