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Fibers of generic maps on surfaces

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ABSTRACT

Using special triangulations of a compact 2-dimensional topological manifold without boundary S, for every closed subset $F \subseteq S$ we construct a dense in the mapping space $C(F, [0, 1])$ family of piecewise linear mappings whose fibers consist of components homeomorphic to subcontinua of the figure eight. The number of fibers with a figure-eight component is evaluated for each such map in the case $F = S$. We then prove that every fiber of a generic map in $C(F, [0, 1])$ consists only of components being either a singleton or a figure-eight-like hereditarily indecomposable continuum. This extends a result of Z. Buczolich and U.B. Darji.

1. Introduction

Given any two metric spaces X and Y, by $C(X, Y)$ we denote the space of all continuous mappings from X into Y with the supremum norm. By saying that *a generic continuous function* $f \in C(X, Y)$ *has property* $\mathcal{P}$ we mean the existence of a dense $\mathbb{G}_\delta$-subset $G \subseteq C(X, Y)$ such that every $f \in G$ has the property $\mathcal{P}$. For compact metric spaces and the unit interval $I = [0, 1]$ with the natural topology, M. Levin proved the following theorem:

Theorem 1.1. *([1]) Let X be a compact metric space. Then every component of every fiber of a generic $f \in C(X, I)$ is a hereditarily indecomposable continuum.*

A *continuum* is a connected compact metric space. We say that a continuum is *indecomposable* if it cannot be represented as a union of its two proper subcontinua and it is *hereditarily indecomposable* if its every subcontinuum is indecomposable.

Independently of Levin, J. Krasinkiewicz obtained a stronger result:

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Theorem 1.2. *([2]) Let X be a compact metric space and M be a manifold of positive dimension. Then every component of every fiber of a generic $f \in C(X,M)$ is a hereditarily indecomposable continuum.*

J. Song and E.D. Tymchatyn generalized the Krasinkiewicz theorem to polygons:

Theorem 1.3. *([3]) Let X be a compact metric space and P be a locally finite polygon. Then every component of every fiber of a generic $f \in C(X,P)$ is a hereditarily indecomposable continuum.*

These three theorems characterize fibers of a generic map between two given spaces in a very general way – they state only that components of these fibers are hereditarily indecomposable continua. A major step toward a more precise description is the paper by Z. Buczolich and U.B. Darji [4], where the characterization is given of components of fibers of a generic map from the 2-dimensional sphere S^2 into the unit interval I in terms of ε-mappings onto *the figure eight* $\mathcal{8}$ – a space homeomorphic to the wedge of two circles $S^1 \vee S^1$. A function $f \in C(X,Y)$ is an *ε-mapping* if the preimage of every point has diameter less than ε. For a finite graph P a continuum X is *P-like* if for every $\varepsilon > 0$ there exists an ε-mapping from X onto P.

Theorem 1.4. *(Buczolich and Darji [4]) Every component of every fiber of a generic $f \in C(S^2, I)$ is either a singleton or an $\mathcal{8}$-like hereditarily indecomposable continuum.*

This paper generalizes the theorem of Buczolich and Darji to any closed subset of a compact surface (Theorem 3.4). The idea of the proof is based on the technique of Buczolich and Darji, i.e. by constructing appropriate triangulations of a surface we prove the existence of a dense family of continuous mappings with components of fibers being points, circles or eights. The existence of this family is crucial for the proof of the main theorem (Lemma 3.3). A very simple method of triangulating S^2 presented in [4] cannot be directly carried over an arbitrary compact surface, so in this paper we propose a new way of constructing triangulations of compact surfaces. We call those triangulations eight-like (cf. Definition 2.1). They have a series of features common with the triangulation presented in [4], hence the proofs of Lemmas 2.3, 2.6, 2.7 and 2.15 are strongly based on the proofs of corresponding lemmas from [4].

2. Triangulations of surfaces

By a *surface* we always mean a 2-dimensional topological manifold, i.e. a non-empty locally Euclidean, second countable, Hausdorff space. By a triangulation $\mathcal{T}$ of a surface S we mean a pair $(\mathcal{K}, \varphi)$ where $\mathcal{K}$ is a Euclidean simplicial complex and $\varphi : |\mathcal{K}| \to S$ is a homeomorphism from the polygon $|\mathcal{K}|$ induced by $\mathcal{K}$ onto S. For simplicity, we identify the complex $\mathcal{K}$ with its polygon $|\mathcal{K}|$. If $v = \varphi(x)$ for some 0-simplex $x \in \mathcal{K}$ ($\approx |\mathcal{K}|$), then v is a *vertex* of $\mathcal{T}$. We define similarly *edges* and *triangles* of $\mathcal{T}$ as images by φ of respectively 1- and 2-simplices from $\mathcal{K}$. If vertices u and v of $\mathcal{T}$ are distinct end points of an edge e of $\mathcal{T}$, then we denote e simply by uv. The sets of vertices and edges of $\mathcal{T}$ are denoted by $\mathcal{V}$ and $\mathcal{E}$, respectively. The set of triangles of $\mathcal{T}$ is identified with $\mathcal{T}$ itself, i.e. $t \in \mathcal{T}$ means that t is a triangle of $\mathcal{T}$. Given a vertex $v \in \mathcal{V}$, we denote by $N(v)$ the set of all neighbors of v in $\mathcal{T}$, i.e. $N(v) = \{w \in \mathcal{V}\colon\ vw \in \mathcal{E}\}$. $V(e)$ and $V(t)$ denote, respectively, the 2-element set of end points of an edge $e \in \mathcal{E}$ and the 3-element set of vertices of a triangle $t \in \mathcal{T}$. Given a graph G, the *degree* of a vertex v of G, denoted by $\deg_G v$, is the cardinality of $N(v)$ in G.

Definition 2.1. A triangulation $\mathcal{T}$ is *eight-like* if every $v \in \mathcal{V}$ has degree greater than 3 and there exists a 3-coloring $c : \mathcal{V} \to \{-, 0, +\}$ of vertices of $\mathcal{T}$ satisfying the following conditions:

1. for every $v \in \mathcal{V}$:
   - if $\deg_{\mathcal{T}} v \neq 4, 6$, then $c(v) \neq 0$;