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Special homeomorphisms and approximation for Cantor systems

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ABSTRACT

E. Akin, E. Glasner, and B. Weiss had constructed the special homeomorphism that has a dense G_δ conjugacy class in the space of all Cantor homeomorphisms. M. Hochman showed that the universal odometer is the special homeomorphism in the space of all topologically transitive Cantor homeomorphism. Following the approach of E. Akin, E. Glasner, and B. Weiss, we show that the universal odometer is the special homeomorphism in the space of all chain transitive Cantor systems. We extend this result to the space of chain transitive systems that are restricted by a periodic spectrum. Further, we construct the special homeomorphism in the space of all chain recurrent systems. In doing so, every 0-dimensional system is described as the inverse limit of a sequence of finite directed graphs and graph homomorphisms. In the previous paper, we had shown that a certain periodic condition determines whether a Cantor system approximates a chain mixing Cantor system by topological conjugacies. We shall extend this result to the chain transitive case. These conditions are described in terms of sequences of finite directed graphs and graph homomorphisms.

1. Introduction

Let $\mathbb{Z}$ denote the set of all integers; $\mathbb{N}$, the set of all non-negative integers; and $\mathbb{N}^+$, the set of all positive integers. Let (X, d) be a compact metric space. Let $\mathbf{H}^+(X) := \{f : X \to X \mid f \text{ is continuous and } f(X) = X\}$. A pair (X, f) $(f \in \mathbf{H}^+(X))$ is called a *topological dynamical system*. We define a metric on $\mathbf{H}^+(X)$ as follows:

$$d(f, g) := \sup_{x \in X} d\bigl(f(x), g(x)\bigr) \quad \text{for all } f, g \in \mathbf{H}^+(X).$$

Then, $(\mathbf{H}^+(X), d)$ is a complete metric space of uniform convergence. Let

$$\mathbf{H}(X) := \bigl\{f \in \mathbf{H}^+(X) \bigm| f \text{ is a homeomorphism}\bigr\}$$

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with the subspace topology on it. Although $\mathbf{H}(X)$ is not complete by the above metric, it is completely metrizable by the metric d' as follows:

$$d'(f,g) := d(f,g) + d\bigl(f^{-1}, g^{-1}\bigr) \quad \text{for all } f,g \in \mathbf{H}(X).$$

A separable completely metrizable space is called a Polish space. Therefore, both $\mathbf{H}^+(X)$ and $\mathbf{H}(X)$ are Polish spaces. Let C be the Cantor set. We mainly consider the case in which X is 0-dimensional. Topological spaces that are homeomorphic to C are characterized as compact 0-dimensional perfect metrizable spaces. A topological dynamical system (X,f) is said to be a *Cantor system* if X is homeomorphic to C. We abbreviate $\mathbf{H}^+(C)$ as $\mathbf{H}^+$ and $\mathbf{H}(C)$ as $\mathbf{H}$. If X is 0-dimensional, we call (X,f) a 0-*dimensional system*. A subset of a topological space is called a G_δ set if it is written as an intersection of at most countably many open sets. In a completely metrizable space, by the Baire category theorem, a subset is a dense G_δ set if and only if it is the intersection of at most countably many dense open subsets. We refer to J.C. Oxtoby [10]. A topological space is called a Baire space if an intersection of countably many dense open sets is dense. In Baire spaces, a subset is said to be *residual* if it contains a dense G_δ set. Suppose that a set of systems forms a Baire space. Then, a property of the system is said to be *generic* if the set of systems that satisfy the property forms a residual subset. By Alexandroff's theorem, in completely metrizable spaces, a subspace is a G_δ set if and only if it is completely metrizable (J.C. Oxtoby [10, Theorems 12.1 and 12.3]).

Let (X,f) and (Y,g) be topological dynamical systems. If there exists a continuous surjective mapping $\phi: Y \to X$ such that $\phi \circ g = f \circ \phi$, then (X,f) is called a *factor* of (Y,g) and ϕ is called a *factor mapping*. If a factor mapping $\phi: Y \to X$ is a homeomorphism, then (X,f) is said to be (*topologically*) *conjugate* (or *isomorphic*) to (Y,g) and ϕ is called a (*topological*) *conjugacy* or an *isomorphism*, in which case $\phi \circ g \circ \phi^{-1} = f$. In [6], A.S. Kechris and C. Rosendal showed that there exists an $f \in \mathbf{H}$ whose conjugacy class is a dense G_δ set in $\mathbf{H}$. Further, in [2], E. Akin, E. Glasner, and B. Weiss constructed this homeomorphism. In [5, Theorem 1.1], M. Hochman showed that the space of transitive systems in $\mathbf{H}$ is Polish and the isomorphism class of the universal odometer is a dense G_δ set in the space. We shall present another proof of this result of M. Hochman in the form of Theorem 1.12, and obtain an extension in the form of Theorem 1.13.

On the other hand, in [11], we introduced the following notion in the case of general Cantor systems:

Definition 1.1. Let (Y,g) and (X,f) be topological dynamical systems. Suppose that there exists a sequence of homeomorphisms $\phi_k: Y \to X$ $(k = 1,2,\ldots)$ such that $\phi_k \circ g \circ \phi_k^{-1} \to f$ as $k \to \infty$. Then, we say that (Y,g) *approximates* (X,f) *by* (*topological*) *conjugacies*, and we write $(Y,g) \rhd (X,f)$.

Before we presented this notion, H. Lin and H. Matui [7] and H. Matui [9] had introduced the notion of weak approximate conjugacy in the case of Cantor minimal systems in the context of C^*-algebras, and they had obtained a related result. Let (Y,g) and (X,f) be Cantor systems such that $(Y,g) \rhd (X,f)$. First, consider the case in which there exists a $y \in Y$ and an $n \in \mathbb{N}^+$ such that $g^n(y) = y$. Then, it is easy to see that there exists an $x \in X$ such that $f^n(x) = x$. Therefore, we introduce the following notation:

Definition 1.2. For a Cantor system (X,f), we define the following:

$$\mathrm{Per}(X,f) := \bigl\{ n \in \mathbb{N}^+ \bigm| \text{there exists an } x \in X \text{ such that } f^n(x) = x \bigr\}.$$

Sometimes, we abbreviate $\mathrm{Per}(X,f)$ as $\mathrm{Per}(f)$.

Remark 1.3. If $n \in \mathrm{Per}(X,f)$, then $nm \in \mathrm{Per}(X,f)$ for all $m \in \mathbb{N}^+$.

From the above argument, $(Y,g) \rhd (X,f)$ implies that $\mathrm{Per}(Y,g) \subseteq \mathrm{Per}(X,f)$. In the previous paper, we obtained the following: