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I-spaces, nodec spaces and compactifications

Karim Belaid [a,*], Lobna Dridi [b]

[a] *University of Dammam, Faculty of Sciences of Dammam, Girls College, Department of Mathematics, PO Box 383, Dammam 31113, Saudi Arabia*
[b] *University of Monastir, Higher Institute of Mathematics and Informatics (ISIM), Department of Mathematics, PO Box 223, 5000 Monastir, Tunisia*

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A B S T R A C T

In this paper, we describe compact nodec spaces and we characterize space such that its one point compactification (respectively Wallman compactification) is nodec. We also establish a characterization of spaces such that their compactification is an I-space. And we give necessary and sufficient conditions on the space X in order to get its Herrlich compactification remainder finite.

Introduction

A subset N of a topological space X is called *nowhere dense* if the interior of the closure of N is the empty set. Recall that a space X is a *nodec space* if each nowhere dense subset of X is closed. A topological space X is an I*-space* if its derived set X^d (that is the set of accumulation points) is discrete. And, if every subset of X is an intersection of a closed subset and an open set of X, then the X is said to be *submaximal*. Submaximal spaces have been studied by several authors (see, for instance, [1,7,9]).

Classically we have the following implications:

$$I\text{-space} \Rightarrow \text{Submaximal} \Rightarrow \text{Nodec}$$

It is shown in [3] that a compactification $K(X)$ of a topological space X is submaximal if and only if for each dense subset D of X, the following properties hold:

* Corresponding author.
E-mail addresses: belaid412@yahoo.fr (K. Belaid), lobna.dridi@gnet.tn (L. Dridi).

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(i) $K(X)\backslash D$ is finite.
(ii) For each $x \in K(X) \setminus D$, $\{x\}$ is closed.

The first section of this paper contains some remarks of nodec spaces and compact nodec spaces. We also give a necessary and sufficient conditions on a space X in order to get its one point compactification (resp. Wallman compactification) nodec.

The second section deals with the characterization of spaces such that their compactification are I-spaces. We establish a necessary and sufficient conditions on a space X to get its one point compactification (resp. Wallman compactification) an I-space.

In the third section we give a characterization of spaces such that their Herrlich compactification remainder is finite.

Throughout this paper we consider spaces on which no separation axioms are assumed unless explicitly stated. Let X be a topological space and A be a subset of X. The closure of A in X is denoted by $cl_X(A)$, and if A is finite we denote $Card(A)$ the cardinality of A.

1. Nodec spaces and compactifications

Let first recall the definition of the Krull dimension of a T_0-space. Let $(X, \mathcal{T})$ be a T_0-space. Then X has a partial order $\leqslant$, induced by $\mathcal{T}$ by taking $x \leqslant y$ if and only if $y \in cl_X(x)$. Hence $cl_X(x) = \{y \mid y \geqslant x\}$ is the specialization of x [8]. The notion of the Krull dimension defined on the prime spectrum of a ring has been generalized to T_0-spaces [4]. The chain $x_0 < x_1 < \cdots < x_n$ of elements of X is said to be a chain of length n. The supremum of the lengths of chains is called the *Krull dimension* of $(X, \mathcal{T})$ and is denoted by $dim_K(X, \mathcal{T})$.

In [2] the authors have proved that if a T_0-space $(X, \mathcal{T})$ is submaximal, then $dim_K(X, \mathcal{T}) \leqslant 1$.

Proposition 1.1. *Let $(X, \mathcal{T})$ be a nodec T_0-space. Then $dim_K(X, \mathcal{T}) \leqslant 1$.*

Proof. Let $x \in X$. If $y \in cl_X(x)\backslash\{x\}$, then $\{y\}$ is a nowhere dense subset of X. Hence $\{y\}$ is closed. Thus y is a maximal point (for the order induced by the topology $\mathcal{T}$) of X. Therefore $dim_K(X, \mathcal{T}) \leqslant 1$. □

Example 1.2. There exists a nodec space with Krull dimension 1 which is not a submaximal space.

Let $\mathbb{Z}$ be the set of the integers, equipped with the topology $\mathcal{T} = \{\emptyset\} \cup \{U \subseteq \mathbb{Z} \mid\ 0 \in U \text{ and } \mathbb{Z}\backslash U \text{ is finite}\}$. Clearly $dim_K(\mathbb{Z}, \mathcal{T}) = 1$, whilst $(\mathbb{Z}, \mathcal{T})$ is nodec, since every nowhere dense set is finite. On the other hand, $(\mathbb{Z}, \mathcal{T})$ is not submaximal, since $\{0\}$ is not an intersection of a closed subset and an open subset of $\mathbb{Z}$.

In fact, more can be said.

Example 1.3. There exists a T_0-space with Krull dimension 1 which is not a nodec space.

Let $X = \mathbb{Z} \cup \{\omega_1, \omega_2\}$ and $\leqslant$ be the order on X defined by $2n+1 \leqslant 2n$, $2n-1 \leqslant 2n$ and $\omega_1 \leqslant \omega_2$. Let $\mathcal{U}$ be a collection of upper subsets X (that is, $A \in \mathcal{U}$ if and only if $\{y \in X \mid x \leqslant y\} \subseteq A$, for each $x \in A$). Let T be the topology on X whose closed sets are empty set and elements A of $\mathcal{A}$ such that A is finite or $\omega_2 \in A$. Let $L = \{2n \mid n \in \mathbb{Z}\}$. Then $cl_X(L) = L \cup \{\omega_2\}$ and the interior of $cl_X(L)$ is the empty set. Hence, L is a nowhere dense subset of X but not a closed set of X. Thus X is not nodec.

We need the following definition to describe compact nodec spaces.

Definition 1.4. A topological space X is said to be a *strong-nodec* space (*s-nodec*, for short) if each nowhere dense subset is finite and closed.