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Hereditarily supercompact spaces [☆]

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ABSTRACT

A topological space X is called *hereditarily supercompact* if each closed subspace of X is supercompact. By a combined result of Bula, Nikiel, Tuncali, Tymchatyn, and Rudin, each monotonically normal compact Hausdorff space is hereditarily supercompact. A dyadic compact space is hereditarily supercompact if and only if it is metrizable. Under (MA+¬CH) each separable hereditarily supercompact space is hereditarily separable and hereditarily Lindelöf. This implies that under (MA+¬CH) a scattered compact space is metrizable if and only if it is separable and hereditarily supercompact. The hereditary supercompactness is not productive: the product $[0,1]\times\alpha D$ of the closed interval and the one-point compactification αD of a discrete space D of cardinality $|D|\geqslant \mathrm{non}(\mathcal{M})$ is not hereditarily supercompact (but is Rosenthal compact and uniform Eberlein compact). Moreover, under the assumption $\mathrm{cof}(\mathcal{M})=\omega_1$ the space $[0,1]\times\alpha D$ contains a closed subspace X which is first countable and hereditarily paracompact but not supercompact.

1. Introduction

This paper is devoted to studying hereditarily supercompact spaces. By definition, a topological space X is *hereditarily supercompact* if each closed subspace Z of X is supercompact.

We recall that a topological space X is called *supercompact* if it has a subbase of the topology such that each cover of X by elements of this subbase has a two-element subcover. By the Alexander Lemma (see e.g. [11, 3.12.2(a)]), each supercompact space is compact. The theory of supercompact spaces was intensively developed in the seventies through the nineties of the twentieth century, see [4,5,7,15–17,22].

According to a result of Strok and Szymański [22], each compact metrizable space is supercompact. Alternative proofs of this important result were given by van Douwen [8], Mills [17] and Dębski [7]. The most general result in this direction was obtained by Bula, Nikiel, Tuncali, Tymchatyn [5] and Rudin [20] who proved that each monotonically normal compact space is supercompact and hence hereditarily supercompact. On the other hand, a dyadic compact space (in particular, a Tychonoff or Cantor cube) is

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hereditarily supercompact if and only if it is metrizable, see Theorem 2.9. In Corollary 2.8 we prove that under (MA+¬CH), each separable hereditarily supercompact space is hereditarily separable and hereditarily Lindelöf. This implies that under (MA+¬CH) a scattered compact space is metrizable if and only if separable and hereditarily supercompact.

In this paper we present several constructions of hereditarily supercompact spaces and shall prove that the hereditary supercompactness is not productive: the closed interval $[0,1]$ and the one-point compactification αD of a discrete space D of cardinality $|D| \geqslant \mathrm{non}(\mathcal{M}) \geqslant \aleph_1$ are hereditarily supercompact but their product $[0,1] \times \alpha D$ is not. Here $\mathrm{non}(\mathcal{M})$ denotes the smallest cardinality of a non-meager subset of the real line. Under certain set-theoretic assumptions (namely, $\mathrm{cof}(\mathcal{M}) = \omega_1$), the product $[0,1] \times \alpha D$ contains a closed subspace X which is first countable, hereditarily paracompact, but not supercompact. Being a closed subspace of $[0,1] \times \alpha D$, the space X is uniform Eberlein and Rosenthal compact.

2. Some properties of hereditarily supercompact spaces

In this section we study the interplay between the class of hereditary supercompact spaces and other classes of compact spaces and present several constructions preserving the hereditary supercompactness. All topological spaces considered in this paper are regular and hence Hausdorff. For a subset A of a topological space X by $\mathrm{cl}(A)$ and $\mathrm{Int}(A)$ we shall denote the closure and interior of A in X, respectively. By $[X]^{<\omega}$ we shall denote the family of all non-empty finite subsets of a set X.

To detect supercompact spaces we shall apply a characterization of the supercompactness in terms of binary closed k-networks.

We recall that a family $\mathcal{K}$ of subsets of a topological space X is called

- *linked* if $A \cap B \neq \emptyset$ for any subsets $A, B \in \mathcal{K}$;
- *binary* if each linked subfamily $\mathcal{L} \subseteq \mathcal{K}$ has non-empty intersection $\bigcap \mathcal{L} \neq \emptyset$;
- a *k-network* if for each open set $U \subseteq X$ and a compact subset $K \subseteq U$ there is a finite subfamily $\mathcal{F} \subseteq \mathcal{K}$ such that $K \subseteq \bigcup \mathcal{F} \subseteq U$;
- a *closed k-network* if $\mathcal{K}$ is a k-network consisting of closed subsets of X.

To detect closed k-networks we shall apply the following simple lemma.

Lemma 2.1. *A family $\mathcal{K}$ of closed subsets of compact space X is a k-network in X if and only if for each point $x \in X$ and its neighborhood $U \subseteq X$ there is a finite subfamily $\mathcal{F} \subseteq \mathcal{K}$ such that $x \in \mathrm{Int}(\bigcup \mathcal{F}) \subseteq \bigcup \mathcal{F} \subseteq U$ and $x \in \mathrm{cl}(\mathrm{Int}(F))$ for all $F \in \mathcal{F}$.*

Proof. The "if" part is trivial. To prove the "only if" part, assume that $\mathcal{K}$ is a k-network in X. Let us fix a point $x \in X$ and its neighbourhood U. By the regularity of X there is an open set V such that $x \in V \subseteq \mathrm{cl}\, V \subseteq U$. The family $\mathcal{K}$, being a k-network, contains a finite subfamily $\mathcal{F} \subseteq \mathcal{K}$ such that $\mathrm{cl}\, V \subseteq \bigcup \mathcal{F} \subseteq U$. In the family $\mathcal{F}$ consider the subfamily $\mathcal{F}_x = \{F \in \mathcal{F}\colon\ x \in \mathrm{cl}(\mathrm{Int}(F))\}$ and observe that the set $V_x = V \setminus \bigcup_{F \in \mathcal{F} \setminus \mathcal{F}_x} \mathrm{cl}(\mathrm{Int}(F))$ is an open neighborhood of x. It remains to prove that $V_x \subseteq \bigcup \mathcal{F}_x$. Assuming the opposite, we would conclude that the open set $W = V_x \setminus \bigcup \mathcal{F}_x$ is non-empty. Since $W \subseteq V_x \subseteq V \subseteq \bigcup \mathcal{F}$, for some set $F \in \mathcal{F} \setminus \mathcal{F}_x$ the intersection $F \cap W$ has non-empty interior in W, which implies that the sets $W \subseteq V_x$ meet the interior of F. But this contradicts the choice of the set V_x. $\square$

The following characterization of supercompactness is well-known for specialists. We present the proof for convenience of the reader.

Theorem 2.2. *A compact Hausdorff space X is supercompact if and only if it possesses a binary closed k-network.*