Topology and its Applications 161 (2014) 330–342

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Bornological modifications of hyperspace topologies

Tom Vroegrijk

ARTICLE INFO

Article history:
Received 21 October 2012
Received in revised form 17 October 2013
Accepted 24 October 2013

Keywords:
Bornology
Bornological convergence
Hyperspace topology
Hausdorff distance
Pretopological space

ABSTRACT

The bornological convergence structures that have been studied recently as generalizations of Attouch–Wets convergence define pretopologies on hyperspaces. In this paper we characterize the topological reflections of these pretopologies and translate the constructions necessary to define bornological convergence to a broader spectrum of hyperspace topologies.

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1. Introduction

Attouch–Wets convergence is a convergence structure on the hyperspace of a metric or normed linear space with various applications in convex analysis. This type of convergence, also called bounded Hausdorff convergence, defines a topology that is coarser than the topology defined by the Hausdorff distance. The metrically bounded subsets play a fundamental role in this construction. Looking at the definition of Attouch–Wets convergence, the question arises whether this idea can be extended by using arbitrary ideals of sets instead of metrically bounded sets. In [6] this generalization of Attouch–Wets convergence was introduced and studied into detail. The construction that was given in this paper yields a neighbourhood filter for each element of the hyperspace 2^X of all subsets of a metric space (X, d), with one setback: this family of neighbourhood filters does in general not generate a topology, i.e. it is not certain that each neighbourhood filter has a base of open sets. Such an assignment of neighbourhood filters to the points of a set is called a *pretopology*.

Determining which conditions on a bornology are necessary and sufficient such that the corresponding pretopology is in fact topological is the main subject of [4]. In this paper it is extensively explained what properties a bornology should have so its bornological convergence structure (or its lower or upper part) be topological.

Within the category of pretopological spaces and continuous maps the topological spaces form a concretely reflective subcategory. This means that for each pretopology on a set X there is a finest topology on X such that a neighbourhood of a point in this topology is also a neighbourhood of that point in the pretopology. This brings us to the first question we want to answer in this paper: what is the topological reflection of a (lower or upper) bornological convergence structure as it was defined in [6]?

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http://dx.doi.org/10.1016/j.topol.2013.10.035

The modifications that are necessary to transform a Hausdorff distance topology to Attouch–Wets convergence or (more general) bornological convergence can be performed on a much broader spectrum of hyperspace topologies. The second goal of this paper is to describe when such a *bornological modification* of a hyperspace topology results in a new hyperspace topology (rather than a pretopology) and to give a description of the topological reflection in case the resulting structure is a pretopology.

2. Preliminaries

2.1. Pretopological spaces

Pretopologies can be characterized in various ways. One way to characterize these objects is by a map from the set X of points to the set of filters on X. The filter thus associated to a point will then be called the *neighbourhood filter* of that point. A map between pretopological spaces is called *continuous* iff the inverse image of a neighbourhood of $f(x)$ is a neighbourhood of x. A set $G \subseteq X$ is called *open* iff it is a neighbourhood of each of its points. Having assigned to each point a neighbourhood filter it is possible to define the closure operator cl as follows

$$a \in \mathrm{cl}(A) \quad \Leftrightarrow \quad X \setminus A \text{ is no neighbourhood of } a.$$

A closure operator that is constructed in this way has three essential properties.

- $A \subseteq \mathrm{cl}(A)$.
- $A \subseteq B \ \Rightarrow\ \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
- $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Conversely, each closure operator with these properties defines a pretopology by assigning to each point x the following neighbourhood filter

$$\{V \subseteq X \mid x \notin \mathrm{cl}(X \setminus V)\}.$$

This correspondence between pretopologies and closure operators is one-to-one. Using closure operators, continuous maps between pretopological spaces can be characterized as maps that send elements of $\mathrm{cl}(A)$ to elements of $\mathrm{cl}(f(A))$ for each $A \subseteq X$. A set G is open in a pretopological space with closure operator cl iff $a \notin \mathrm{cl}(X \setminus G)$ for each $a \in G$. This is equivalent to $G = X \setminus \mathrm{cl}(X \setminus G)$.

It is clear that each topology also defines a pretopology. The only difference between both being that in a topological space each neighbourhood filter has a base of open sets. In terms of closure operators we can say that a pretopology is a topology iff its closure operator is idempotent, i.e.

- $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

Throughout this text the closure operator associated with a topology T will be denoted as cl^T. Although not every pretopology is a topology, we do have that the category of topological spaces and continuous maps is a concretely reflective subcategory of the category of pretopological spaces. For more information on this categorical terminology we refer the reader to [1] and [7]. In a sense, we have that for each pretopology there is a topology that is the *most similar* to it. Concretely, it means that for each pretopological space X there is a topological space X^τ with the same underlying set such that whenever

$$f : X \to Y$$

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