Topology and its Applications 161 (2014) 354–363

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A note on products of (weakly) discretely generated spaces ☆

Liang-Xue Peng *, Chong Yang

College of Applied Science, Beijing University of Technology, Beijing 100124, China

A R T I C L E   I N F O

Article history:
Received 25 May 2013
Received in revised form 11 October 2013
Accepted 25 October 2013

MSC:
primary 54D99
secondary 54B10

Keywords:
Discretely generated space
Weakly discretely generated space
l-Nested space
Generalized ordered space

A B S T R A C T

In the first part of this note, we mainly get the following conclusions. If X is a regular space with a nested local base at a point a of X and Y is discretely generated at a point b of Y, then $X \times Y$ is discretely generated at $\langle a, b\rangle$. We finally show that if X is a GO-space and Y is a discretely generated space then $X \times Y$ is discretely generated.

In the last part of this note, we mainly get the following conclusions. If X is a locally compact GO-space and there exists a closed discrete set F such that for any point $x \in X \setminus F$, there is a neighborhood V_x of x in X such that $V_x \times Y$ is weakly discretely generated, where Y is a weakly discretely generated space, then $X \times Y$ is weakly discretely generated. If X is a locally compact GO-space such that $X \times Y$ is weakly discretely generated and $lX \setminus X$ is scattered, where lX is a linearly ordered compactification of X and Y is a weakly discretely generated space, then $lX \times Y$ is weakly discretely generated.

1. Introduction

Recall that a space X is *discretely generated at a point* $x \in X$ if for any $A \subseteq X$ with $x \in \overline{A}$, there exists a discrete set $D \subseteq A$ such that $x \in \overline{D}$. The space X is *discretely generated* if it is discretely generated at every point $x \in X$. A space X is *weakly discretely generated* if for any non-closed set $A \subseteq X$, there exists a discrete set $D \subseteq A$ such that $\overline{D} \setminus A \neq \emptyset$ [4]. Obviously a discretely generated space is weakly discretely generated.

Discretely generated spaces were introduced in [4] where it was established, among many other results, that every compact Hausdorff space of countable tightness is discretely generated as well as any monotonically normal space and any regular space with a nested local base at every point. Many of these results were generalized in [3]. In [8], Ivanov and Osipov constructed, under CH, an example of a compact discretely generated space X such that $X \times X$ is not discretely generated. Recall that a chain $\mathcal{V}$ consisting of open subsets of a space X is called a *nest* in X. A space X is called *nested* [1] if for every point $x \in X$ there exists a nest in X which is a local base of x in X. A nested space is also called a *l-nested* space in [4] and [13].

☆ Research supported by the National Natural Science Foundation of China (Grant No. 11271036).
* Corresponding author.
E-mail addresses: pengliangxue@bjut.edu.cn (L.-X. Peng), yangchong@emails.bjut.edu.cn (C. Yang).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.10.038

It was proved in [4, Theorem 3.13] that every regular l-nested space is discretely generated. In [13], it was shown that finite products of Hausdorff l-nested spaces and arbitrary box products of monotonically normal spaces are discretely generated.

The following conclusions were proved in [2].

(1) If X is a regular space with a nested local base at a and Y is regular and discretely generated at b and $t(b,Y) < \chi(a,X)$, then $X \times Y$ is discretely generated at $\langle a,b\rangle$ [2, Theorem 2.1].
(2) The product of a regular discretely generated space X with a first countable regular space Y is discretely generated [2, Theorem 2.3].
(3) If X is a regular space with a nested local base at a and Y is regular and discretely generated at b and $\chi(a,X) = \chi(b,Y)$, then $X \times Y$ is discretely generated at $\langle a,b\rangle$ [2, Theorem 2.4].

In this note, we show that if X is a regular space with a nested local base at a point a of X and Y is discretely generated at a point b of Y, then $X \times Y$ is discretely generated at $\langle a,b\rangle$. Thus the above conclusions which appear in [2] are generalized. We also show that if X is a GO-space and Y is a discretely generated space then $X \times Y$ is discretely generated.

In [2], it was proved that if Y is weakly discretely generated and X is a finite product of ordinals, then $X \times Y$ is weakly discretely generated. In the last part of this note, we firstly get the following result. Let $(X, \leqslant)$ be a compact linearly ordered topological space such that $a = \min X$ and $b = \max X$ and let Y be a weakly discretely generated space. If $[a,c] \times Y$ is weakly discretely generated for each $c < b$, where $[a,c] = \{x \in X\colon\ a \leqslant x \leqslant c\}$, then $X \times Y$ is weakly discretely generated. We finally prove that if X is a locally compact GO-space such that $X \times Y$ is weakly discretely generated and $lX \setminus X$ is scattered, where lX is a linearly ordered compactification of X and Y is a weakly discretely generated space, then $lX \times Y$ is weakly discretely generated.

The following notions can be found in [10]. A *linearly ordered topological space* (abbreviated LOTS) is a triple $\langle X, \lambda, \leqslant\rangle$ where $\langle X, \leqslant\rangle$ is a linearly ordered set and λ is the usual interval topology defined by $\leqslant$. A *generalized ordered space* (abbreviated GO-space) is a triple $\langle X, \tau, \leqslant\rangle$, where $\langle X, \leqslant\rangle$ is a linearly ordered set and τ is a topology on X that contains the usual interval topology defined by $\leqslant$ and has a base consisting of order-convex sets. A LOTS $Y = \langle Y, \lambda, \leqslant_Y\rangle$ is said to be a *linearly ordered extension* of the GO-space $X = \langle X, \tau, \leqslant_X\rangle$ if $X \subset Y$, $\tau = \lambda|X$ and $\leqslant_X = \leqslant_Y |X$. Furthermore if $\overline{X} = Y$ and the space Y is compact then Y is said to be a *linearly ordered compactification* of X.

For any x in any linearly ordered set X, *the cofinality of x in X*, denoted by $cf(x)$, is the least cardinal κ such that $(\leftarrow, x)$ has a cofinal subset of size κ. Note that if x has an immediate predecessor in X, then $cf(x) = 1$. *The coinitiality of x in X*, denoted by $ci(x)$, is analogously defined. The character of x in $A \subseteq X$ will be denoted by $\chi(x, A)$.

If $x \in X \times Y$, then we denote $x = \langle a,b\rangle$ for some $a \in X$ and $b \in Y$. The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [5].

2. Main results

Theorem 1. *If X is a regular space with a nested local base at a point a of X and Y is discretely generated at a point b of Y, then $X \times Y$ is discretely generated at $\langle a,b\rangle$.*

Proof. If the point a is an isolated point of X, then it is obvious that $X \times Y$ is discretely generated at $\langle a,b\rangle$. So we assume that a is not an isolated point of X. Suppose $\chi(a,X) = \Lambda$. Thus Λ is an infinite regular cardinal. Let $\{V_\alpha\colon\ \alpha < \Lambda\}$ be a nested local base of the point a in X. Since X is regular, we can assume without loss of generality that $\overline{V_\beta} \subseteq V_\alpha$ if $\alpha < \beta < \Lambda$. Define $Z = X \times Y$. Let $A \subseteq Z$ be such that $\langle a,b\rangle \in \overline{A}$. If $\langle a,b\rangle \in A$, then we let $D = \{\langle a,b\rangle\}$. So we assume that $\langle a,b\rangle \notin A$. If $\langle a,b\rangle \in \overline{A \cap (\{a\} \times Y)}$,