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Smooth embeddings of rational homology balls

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The rational homology balls B_n appeared in Fintushel and Stern's rational blow-down construction [3] and were subsequently used in [10,4] to construct exotic smooth manifolds with small Euler numbers. We show that a large class of smooth 4-manifolds have all the B_ns for odd $n \geqslant 3$ embedded in them. In particular, we give explicit examples, using Kirby calculus, of several families of smooth embeddings of the rational homology balls B_n.

1. Introduction

In 1997, Fintushel and Stern [3] defined the rational blow-down operation for smooth 4-manifolds, a generalization of the standard blow-down operation. For smooth 4-manifolds, the standard blow-down is performed by removing a neighborhood of a sphere with self-intersection (-1) and replacing it with a standard 4-ball B^4. The rational blow-down involves replacing a negative definite plumbing 4-manifold with a rational homology ball. In order to define it, we first begin with a description of the negative definite plumbing 4-manifold C_n, $n \geqslant 2$, as seen in Fig. 1, where each dot represents a sphere, S_i, in the plumbing configuration. The integers above the dots are the self-intersection numbers of the plumbed spheres: $[S_1]^2 = -(n+2)$ and $[S_i]^2 = -2$ for $2 \leqslant i \leqslant n-1$.

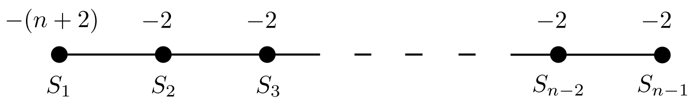

Fig. 1. Plumbing diagram of C_n, $n \geqslant 2$.

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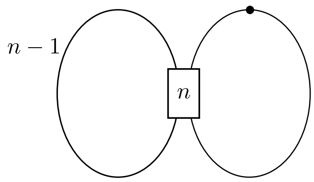

Fig. 2. Kirby diagram of B_n.

The boundary of C_n is the lens space $L(n^2, n-1)$, thus $\pi_1(\partial C_n) \cong H_1(\partial C_n; \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$. (Note, when we write the lens space $L(p, q)$, we mean it is the 3-manifold obtained by performing $-\frac{p}{q}$ surgery on the unknot.) This follows from the fact that $[-n-2, -2, \ldots -2]$, with $(n-2)$ many (-2)s is the continued fraction expansion of $\frac{n^2}{1-n}$.

Let B_n be the 4-manifold as defined by the Kirby diagram in Fig. 2 (for a more extensive description of B_n, see Section 2). The manifold B_n is a rational homology ball, i.e. $H_*(B_n; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q})$. The boundary of B_n is also the lens space $L(n^2, n-1)$ [2]. Moreover, any self-diffeomorphism of ∂B_n extends to B_n [3]. Now, we can define the rational blow-down of a 4-manifold X below in Definition 1.1. Fintushel and Stern [3] also showed how to compute Seiberg–Witten and Donaldson invariants of $X_{(n)}$ from the respective invariants of X. In addition, they showed that certain smooth logarithmic transforms can be alternatively expressed as a series of blow-ups and rational blow-downs. The rational blow-down was particularly useful in constructing 4-manifolds with exotic smooth structures, with small Euler numbers.

Definition 1.1. ([3], also see [7]) Let X be a smooth 4-manifold. Assume that C_n embeds in X, so that $X = C_n \cup_{L(n^2,n-1)} X_0$. The 4-manifold $X_{(n)} = B_n \cup_{L(n^2,n-1)} X_0$ is by definition the *rational blow-down* of X along the given copy of C_n.

One can define the (smooth) *rational blow-up* operation in a similar manner: if there exists a smoothly embedded B_n in a 4-manifold X then one can rationally blow-up X by removing the B_n and gluing in the C_n, along the lens space $L(n^2, n-1)$. Consequently, one can ask: which 4-manifolds can be smoothly rationally blown up? Equivalently, which 4-manifolds contain a smoothly embedded rational homology ball B_n? We prove the following results regarding smooth embeddings of the rational homology balls B_n:

Theorem 1.2. *Let V_{-n-1} be a neighborhood of a sphere with self-intersection number $(-n-1)$. There exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-n-1}$, for all $n \geqslant 2$.*

Theorem 1.3. *Let V_{-4} be a neighborhood of a sphere with self-intersection number (-4). For all $n \geqslant 3$ odd, there exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-4}$. For all $n \geqslant 2$ even, there exists an embedding of the rational homology balls $B_n \hookrightarrow B_2 \# \overline{\mathbb{C}P^2}$.*

Theorems 1.2 and 1.3 above show that there is little obstruction to *smoothly* embedding the rational homology balls B_n into a smooth 4-manifold. In particular, Theorem 1.3 implies that if a smooth 4-manifold X contains a sphere with self-intersection (-4), then one can smoothly embed the rational homology balls B_n into X for all odd $n \geqslant 3$.

One of the implications of Theorem 1.3 is that for a given smooth 4-manifold X, there does not exist an N, such that for all $n \geqslant N$ one cannot find a smooth embedding $B_n \hookrightarrow X$. In the setting of this sort in algebraic geometry, for rational homology ball smoothings of certain surface singularities, such a bound on n does exist, in terms of (c_1^2, χ_h) invariants of an algebraic surface [8,12].

In Section 2 we give a brief description of the rational homology balls B_n. In Section 3 we describe some previously known embeddings of B_n in order to illustrate the differences between them and those