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Shore points of a continuum ☆

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In this paper we extend the study of shore points to every continuum and we prove that every continuum has at least two shore points. This result generalizes an important theorem in Continuum Theory: Every continuum has at leasts two non-cut points. We show that every point of irreducibility is a shore point, and we give some conditions under which a shore point is a point of irreducibility. Also, we characterize uniquely irreducible continua, an arc and a simple closed curve using shore points. Finally, we show that the union of shore points in a uniquely irreducible continua is a shore set.

1. Introduction

The concept of *shore point* was introduced by L. Montejano and I. Puga in [9] to describe the conical pointed property for the hyperspace of subcontinua of a smooth dendroid. Many properties related with these points have been studied in dendroids and λ dendroids, see [2,8,9,13,14,11] and [12].

In this paper we consider shore points not only in dendroids but in any non-degenerate continuum. We prove that every continuum has at least two shore points. This result generalizes a classical result in Continuum Theory: Every continuum has at least two non-cut points [10]. We also prove that every point of irreducibility of X is a shore point, the other implications are not always true. We give some conditions to ensure that a shore point is a point of irreducibility of a continuum, in the same way we show which are the shore points in a finitely irreducible continuum and in a uniquely irreducible continuum.

In the history of Continuum Theory we find many characterizations of an arc. In this paper we prove the following two characterization of an arc: A continuum X is an arc if and only if every subcontinuum of X has only two shore points and X is an arc if and only if X is semi aposyndetic and has only two shore

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points. We characterize all the continua with at least two shore points, and we give a characterization of a simple closed curve.

L. Montejano and I. Puga, [13], proved that the finite union of shore points is a shore set in a smooth dendroid. Later A. Illanes [3] proved that the finite union of disjoint shore continua is a shore set in a dendroid.

2. Preliminaries

A *continuum* is a compact, connected metric space. A *subcontinuum* of a continuum X is a continuum contained in X.

If (X,d) is a continuum, $C(X)$ denotes the hyperspace of subcontinua of X with the Hausdorff metric H defined as follows:

$$H(C,A)=\max\bigl\{\sup\bigl\{d(a,C):\ a\in A\bigr\},\sup\bigl\{d(c,A):\ c\in C\bigr\}\bigr\}.$$

Let X be a continuum, and $p\in X$. The *composant of p in X* is the union of all proper subcontinua of X containing p. It is known that each composant of X is dense in X.

A continuum X is *decomposable* if it can be written as the union of two proper subcontinua. We say that X is *hereditarily decomposable* if each non-degenerate subcontinuum of X is decomposable.

A non-empty subset A of a continuum X is a *shore set* if for every $\varepsilon>0$ there is a subcontinuum C of X such that $A\cap C=\emptyset$ and $H(X,C)<\varepsilon$.

In particular, when $A=\{p\}\subset X$, we say that the point p is a *shore point* of X.

It is easy to see that in the unit interval, end points are shore points, and every point of the Knaster continuum is a shore point.

Every subcontinua of a Knaster continuum is a shore set.

A non-empty subset A of a continuum X is a *shore continuum* if A is a continuum and a shore set.

A point p of a continuum X is not a *cut point* if $X\setminus\{p\}$ is connected.

A point p of a continuum X is a *cut point* if $X\setminus\{p\}$ is not connected.

Lemma 1. *Let X be a continuum and p a shore point of X. Then p is not a cut point.*

Proof. If a point p is a cut point, then $X\setminus\{p\}=H\cup K$ where H and K are open and non-empty subsets of X, so it is clear that p cannot be a shore point. □

We observe that in a continuum X if $p\in X$ is not a cut point, then p is not necessarily a shore point. (See Fig. 1.)

A continuum X is *irreducible between the points a and b* if no proper subcontinuum of X contains these two points. The points a and b are points of irreducibility of X.

A continuum X is *irreducible* if it is irreducible between two of its points.

A continuum X is of type A provided that it is irreducible and it admits an upper semi continuous decomposition $\boldsymbol{D}$ ($\boldsymbol{D}$ is called *admissible*) such that there is a monotone map $G:X\to[0,1]$, $\boldsymbol{D}=\{G^{-1}(t):\ t\in[0,1]\}$ that satisfies:

(1) X is irreducible between any point in $G^{-1}(0)$ and any point in $G^{-1}(1)$.
(2) For every point $x\in X$ if we denote $G^{-1}(G(x))=T_x$, then $Int(T_x)=\emptyset$ and $T_x\in C(X)$.
(3) For every pair of points $x,y\in X$, we have that $T_x=T_y$, or $T_x\cap T_y=\emptyset$.
(4) For every point $x\in X$, $X\setminus T_x$ is not connected.