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Some generalizations of Back's Theorem

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ABSTRACT

The problem of the existence of jointly continuous utility functions is studied. A continuous representation theorem of Back [1] gives the existence of a continuous map from the space of total preorders topologized by closed convergence (Fell topology) to the space of utility functions with different choice sets (partial maps) endowed with a generalization of the compact-open topology. The commodity space is locally compact and second countable. Our results generalize Back's Theorem to non-metrizable commodity spaces with a family of not necessarily total preorders. Precisely, we consider regular commodity spaces having a weaker locally compact second countable topology.

1. Introduction

In this paper we study the utility representation problem with respect to a family of not necessarily total preorders. We deal with the existence of utility functions that depend continuously on the consumer, the consumption set and the preference relation. In the literature this is known as the problem of the existence of the so-called *jointly continuous utility functions*.

Let S be a family of preorders on a topological space (X, τ). The problem is to find a suitable topology on S and topological conditions on X ensuring the existence of a continuous function $u : \mathrm{S} \times X \to \mathbb{R}$ such that for every $\preccurlyeq \in \mathrm{S}$ $u(\preccurlyeq, \cdot)$ is a utility function for $\preccurlyeq$.

Of course, the topology on S should satisfy the following natural condition (*closure condition*): if $x_n \to x$, $y_n \to y$ in X and $\preccurlyeq_n \to \preccurlyeq$ with $x_n \preccurlyeq_n y_n$ for every $n \in \mathbb{N}$ then $x \preccurlyeq y$. Note that if the spaces S and X are metrizable, this condition is equivalent to require the set $\{(\preccurlyeq, x, y)\colon\ x \preccurlyeq y\}$ to be closed in $\mathrm{S} \times X \times X$.

In the second half of the past century this problem was widely studied in the literature. An interesting survey of this argument can be found in [4]. We quote [3,8,9,12,14] for the case of total preorders.

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In [13] Levin proved a general result on the existence of jointly continuous utility functions by assuming that S is metrizable satisfying the "closure condition" and X is locally compact and second countable. Levin's result can be extended to a space $\mathcal{P}$ of closed preorders defined on (closed) subsets $D \subset X$ (see Corollary 1 in [13]). Indeed, put $D = D(\preccurlyeq)$ and $\Phi = \{(\preccurlyeq, x)\colon \preccurlyeq \in \mathcal{P},\ x \in D(\preccurlyeq)\}$, Levin proved that if $M = \{(\preccurlyeq, x, y)\colon \preccurlyeq \in \mathcal{P},\ x, y \in D(\preccurlyeq),\ x \preccurlyeq y\}$ is closed in $\mathcal{P} \times X \times X$, with $\mathcal{P}$ metrizable and X locally compact second countable, there exists a jointly continuous function $u : \Phi \to \mathbb{R}$.

Back in [1] revisited Levin's Theorems using partial maps and hypertopologies.

He considered the space $\mathcal{P}_{\mathcal{L}}$ of total closed preorders defined on closed subsets of a locally compact and second countable space X, endowed with the Fell topology $F(\tau \times \tau)$. He also introduced the space $\mathcal{U}$ of all continuous real partial maps defined on closed subsets of X with the τ_c topology, a generalization of the compact-open topology. He proved the existence of a continuous map $\nu_{\mathcal{L}} : \mathcal{P}_{\mathcal{L}} \to \mathcal{U}$ such that $\nu_{\mathcal{L}}(\preccurlyeq)$ is a utility function for every $\preccurlyeq \in \mathcal{P}_{\mathcal{L}}$. Any such map $\nu_{\mathcal{L}}$ is showed to be a homeomorphism of $\mathcal{P}_{\mathcal{L}\text{-lns}}$ onto $\nu_{\mathcal{L}}(\mathcal{P}_{\mathcal{L}\text{-lns}})$, where $\mathcal{P}_{\mathcal{L}\text{-lns}}$ is the family of total locally non-satiated preorders.

This paper relies on and, in the same time, generalizes Back's result. First, we have pointed out that, using a similar proof, it is possible to prove the existence of a continuous map $\nu : \mathcal{P} \to \mathcal{U}$, where the preorders are not necessarily total.

The hypothesis of totalness cannot be dropped to show that ν is a homeomorphism of $\mathcal{P}_{\text{lns}}$ onto $\nu(\mathcal{P}_{\text{lns}})$. We have proved that such a homeomorphism can be extended to a superset $\mathcal{P}'$ of $\mathcal{P}_{\mathcal{L}\text{-lns}}$.

In [7] Levin's Theorem has been generalized to non-metrizable case. The existence of a jointly continuous utility function is proved under the assumptions that X is a hemicompact, submetrizable k-space and S is metrizable or both S and X are hemicompact, submetrizable and $\mathrm{S} \times X$ is a k-space. Unfortunately, when X is a hemicompact, submetrizable k-space but non-metrizable, then $(CL(X \times X), F(\tau \times \tau))$ is not submetrizable [11].

In this paper, under the hypotheses that X is a regular space, having a weaker locally compact second countable topology η, we have associated to the space of preorders $\mathcal{P}$ a suitable space of preorders $\widetilde{\mathcal{P}}$ satisfying the hypotheses of Back's Theorem for non-total preorders. So, we have proved the existence of a continuous map $\nu : \mathcal{P} \to \mathcal{U}$ such that $\nu(\preccurlyeq)$ is an isotone function for every $\preccurlyeq \in \mathcal{P}$, which is a utility function if $\preccurlyeq$ is an η-closed preorder. The topology on $\mathcal{U}$ is just the τ_c topology, while the topology in $\mathcal{P}$ is a suitable weak-topology that coincides with the topology of closed convergence when X is locally compact second countable.

In the case of compact preorders this topology is weaker than the supremum of $F(\tau \times \tau)$ and $F(\eta \times \eta)$, well known in the literature [2].

2. Notation and preliminaries

Let (X, τ) be a Hausdorff topological space and $CL((X, \tau))$ the set of all non-empty closed subsets of X. We will denote by $F(\tau)$ the Fell topology on $CL((X, \tau))$, that is the topology having as a subbase all sets of the form

$$U^- = \{B \in CL((X, \tau))\colon B \cap U \neq \emptyset\}, \quad U \in \tau \quad \text{and}$$
$$(K^c)^+ = \{B \in CL((X, \tau))\colon B \cap K = \emptyset\}, \quad K \text{ compact in } (X, \tau).$$

The Fell topology is also called the topology of closed convergence (see [4, Definition 8.2.6]).

In Back's paper [1], (X, τ) is a second countable locally compact Hausdorff space. It is well known that these spaces are precisely those ones that have a compatible boundedly compact metric (see [15]).

If (X, τ) is locally compact Hausdorff then the Fell topology is locally compact Hausdorff and it is compatible with classical Kuratowski convergence of nets of closed sets. Further, when (X, τ) is in addition second countable then the Fell topology is also second countable (see [2]).