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On base dimension-like functions of the type Ind

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ABSTRACT

In [5] base dimension-like functions of the type Ind were introduced. These functions were studied only with respect to the property of universality. Here, we first compare these dimensions with the classical large inductive dimension Ind and then study these functions with respect to other standard properties of dimension theory.

1. Introduction and preliminaries

We denote by ω the first infinite cardinal and by $\mathcal{O}$ the class of all ordinals. We also consider two extra symbols, "-1" and "∞" such that $-1 < \alpha < \infty$ for every $\alpha \in \mathcal{O}$, $-1(+)\alpha = \alpha(+)(-1) = \alpha$ for every $\alpha \in \mathcal{O} \cup \{-1, \infty\}$, and $\infty(+)\alpha = \alpha(+)\infty = \infty$ for every $\alpha \in \mathcal{O} \cup \{\infty\}$, where by $(+)$ we denote the natural sum of Hessenberg (see [6]). We recall some properties of natural sum. Let α and β be ordinals. Then,

(1) $\alpha(+)\beta = \beta(+)\alpha$,
(2) if $\alpha_1 < \alpha_2$, then $\alpha_1(+)\beta < \alpha_2(+)\beta$, and
(3) $\alpha(+)n = \alpha + n$ for $n < \omega$.

Let Q be a subset of a space X. We denote by $\mathrm{Cl}_X(Q)$ and $\mathrm{Bd}_X(Q)$ the closure and the boundary of Q in X, respectively.

By a *class of subsets* we mean a class consisting of pairs (Q, X), where Q is a subset of a space X. By a *class of bases* we mean a class consisting of pairs (B, X), where B is a base of a space X.

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Let B be a base of a space X. The minimal ring of subsets of X containing B, that is the smallest family of subsets of X closed under finite unions and finite intersections containing B, is denoted by $B^{\diamondsuit}$.

The large inductive dimension of a space X (see for example [3] and [7]), denoted by $\mathrm{Ind}(X)$, is defined as follows:

(i) $\mathrm{Ind}(X) = -1$ if and only if $X = \emptyset$.
(ii) $\mathrm{Ind}(X) \leqslant \alpha$, where $\alpha \in \mathcal{O}$, if and only if for every pair (K, V) of subsets of X, where K is closed, V is open, and $K \subseteq V$, there exists an open set W of X such that $K \subseteq W \subseteq V$ and $\mathrm{Ind}(\mathrm{Bd}_X(W)) < \alpha$.
(iii) $\mathrm{Ind}(X) = \infty$ if and only if the inequality $\mathrm{Ind}(X) \leqslant \alpha$ does not hold for every $\alpha \in \mathcal{O} \cup \{-1\}$.

A topological space X is called a T_4-space if for every pair (K, F) of disjoint closed subsets of X there exist open subsets U and V of X such that $K \subseteq U$, $F \subseteq V$, and $U \cap V = \emptyset$. We note that a space X is T_4 if and only if for every pair (K, V) of subsets of X, where K is closed, V is open, and $K \subseteq V$, there exists an open subset U of X such that $K \subseteq U \subseteq \mathrm{Cl}_X(U) \subseteq V$.

The base dimension-like functions $\mathrm{b}^{\vee}$-Ind, b-Ind, and $\mathrm{b}^{\diamondsuit}$-Ind of the type Ind were introduced by S.D. Iliadis in [5]. In Section 2 we investigate the relations between of them and we compare these dimensions with the classical large inductive dimension Ind. In Sections 3, 4, and 5 we present for these functions subspace, partition, and sum theorems. Finally, in Section 6 we give some questions concerning these functions.

About some other base dimension-like functions of the type Ind see for example [1,4], and [5].

2. On the base dimension-like functions $\mathrm{b}^{\vee}$-Ind, b-Ind, and $\mathrm{b}^{\diamondsuit}$-Ind

Definition 2.1. (See [5].) Let B be a base for a space X. A pair (K, V) of subsets of X is said to be *$B^{\vee}$-proper* if K is the closure of the finite union of elements of B, V is the finite union of elements of B, and $K \subseteq V$.

Definition 2.2. (See [5].) We denote by $\mathrm{b}^{\vee}$-Ind the base dimension-like function with domain the class of all bases and range the set $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

(i) $\mathrm{b}^{\vee}\text{-Ind}(B, X) = -1$ if and only if $X = \emptyset$.
(ii) $\mathrm{b}^{\vee}\text{-Ind}(B, X) \leqslant \alpha$, where $\alpha \in \mathcal{O}$, if and only if for every $B^{\vee}$-proper pair (K, V) of subsets of X there exists an open subset W of X such that $K \subseteq W \subseteq V$ and $\mathrm{b}^{\vee}\text{-Ind}(\{\mathrm{Bd}_X(W) \cap U\colon U \in B\}, \mathrm{Bd}_X(W)) < \alpha$.

Definition 2.3. (See [5].) Let B be a base for a space X. A pair (K, V) of subsets of X is said to be *B-proper* if K is the closure of an element of B, $V \in B$, and $K \subseteq V$.

Definition 2.4. (See [5].) We denote by b-Ind the base dimension-like function with domain the class of all bases and range the set $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:

(i) $\text{b-Ind}(B, X) = -1$ if and only if $X = \emptyset$.
(ii) $\text{b-Ind}(B, X) \leqslant \alpha$, where $\alpha \in \mathcal{O}$, if and only if for every B-proper pair (K, V) of subsets of X there exists an open subset W of X such that $K \subseteq W \subseteq V$ and $\text{b-Ind}(\{\mathrm{Bd}_X(W) \cap U\colon U \in B\}, \mathrm{Bd}_X(W)) < \alpha$.

Definition 2.5. (See [5].) Let B be a base for a space X. A pair (K, V) of subsets of X is said to be *$B^{\diamondsuit}$-proper* if K is the closure of an element of $B^{\diamondsuit}$, $V \in B^{\diamondsuit}$, and $K \subseteq V$.

Definition 2.6. (See [5].) We denote by $\mathrm{b}^{\diamondsuit}$-Ind the base dimension-like function with domain the class of all bases and range the set $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions: