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On monotone stability ☆

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ABSTRACT

The notion of monotonically monolithic space was introduced by V.V. Tkachuk in 2009 [8]. In this paper we introduce the notion of monotone stability and show that a space $C_p(X)$ is monotonically monolithic if and only if X is monotonically stable. As a consequence, a space $C_p(X)$ is monotonically stable if and only if X is monotonically monolithic. We also prove that $C_p(X)$ is monotonically monolithic when X is a Σ_κ-product of a family of Lindelöf Σ-spaces. These results answer some questions posed by Á. Tamariz-Mascarúa and the author in a previous paper [7].

1. Introduction

A space X is monolithic if $\operatorname{nw}(\operatorname{cl} A) \leqslant \max\{|A|, \omega\}$ for every subspace A of X. This concept was introduced by A.V. Arhangel'skii in [2]. V.V. Tkachuk introduced the concept of a monotonically monolithic space:

Definition 1.1. Given a subset A of a space X, we say that a family $\mathcal{N}$ of subsets of X is an *external network* of A in X if for each $x \in A$ and each open subset U of X with $x \in U$, there is $N \in \mathcal{N}$ such that $x \in N \subset U$.

Definition 1.2. We say that a space X is *monotonically monolithic* [8] if for each $A \subset X$, we can assign an external network $\mathcal{O}(A)$ of $\operatorname{cl}(A)$ in X in such a way that the following conditions hold:

(1) $|\mathcal{O}(A)| \leqslant \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$, then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if $\{A_\alpha\colon \alpha < \gamma\}$ is a family of subsets of X with $A_\alpha \subset A_\beta$ for $\alpha < \beta$, then $\mathcal{O}(\bigcup\{A_\alpha\colon \alpha < \gamma\}) = \bigcup\{\mathcal{O}(A_\alpha)\colon \alpha < \gamma\}$.

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Furthermore, for an infinite cardinal κ, X is said to be *monotonically κ-monolithic* [1] if $\mathcal{O}(A)$ is defined for all sets A with $|A| \leqslant \kappa$ and satisfies the above conditions.

The class of monotonically monolithic spaces turned out to be important. V.V. Tkachuk showed that each element in this class has the D-property, hereditarily [8]. The class of monotonically monolithic spaces is reasonably wide; indeed, this contains the spaces with a point countable base [8], the stratifiable spaces [5,6] and the spaces $C_p(X)$ when X is a Lindelöf Σ-space [8]. In particular, every metrizable space is monotonically monolithic. Among its categorical properties, it is known that monotone monolithicity is preserved by countable products, arbitrary subspaces and closed maps [8].

It is known that $C_p(X)$ is monolithic if and only if X is a stable space, and that $C_p(X)$ is stable if and only if X is a monolithic space [3]. Moreover, $C_p(X)$ is monotonically monolithic for several spaces that are stable; indeed, this happens when X is a Σ_κ-product (hence, a product, σ-product or Σ-product) of spaces with countable network weight and when X is a product of Lindelöf Σ spaces [7]. For this reason in [7] the following questions appear:

Problem 1.3. Given a class $\mathcal{C}$ of topological spaces, determine a topological property $\mathcal{P}(\mathcal{C})$ which satisfies: for every space $X \in \mathcal{C}$, $C_p(X)$ is monotonically monolithic if and only if X has property $\mathcal{P}(\mathcal{C})$.

Question 1.4. Let X be a Σ_κ-product of Lindelöf Σ-spaces. Is $C_p(X)$ then monotonically monolithic?

We introduce the concept of monotonically stable space and show that $C_p(X)$ is monotonically monolithic if and only if X is monotonically stable. As a consequence, a space $C_p(X)$ is monotonically stable if and only if X is monotonically monolithic. So, any Σ_κ-product (hence, any product, σ-product or Σ-product) of spaces with countable network weight is monotonically stable as well as any product of Lindelöf Σ spaces. Furthermore, we prove that if X is a Σ_κ-product of a family of Lindelöf Σ-spaces then X is monotonically stable, that is, $C_p(X)$ is monotonically monolithic. This answers Question 1.4 in the affirmative.

2. Notation and terminology

Every space in this article is a Tychonoff space with more than one point. The letters α, β and γ represent ordinal numbers and the letters κ and λ represent infinite cardinal numbers; ω is the first infinite cardinal.

Given two families $\mathcal{N}$ and $\mathcal{C}$ of subsets of a space X, we say that $\mathcal{N}$ is a *network modulo* $\mathcal{C}$ if for any $C \in \mathcal{C}$ and any open set $U \subset X$ with $C \subset U$ there is $N \in \mathcal{N}$ such that $C \subset N \subset U$. A space X is *Lindelöf Σ* if there exist a compact cover $\mathcal{K}$ of the space X and a countable family $\mathcal{N}$ of subsets of X which is a network modulo $\mathcal{K}$. A family $\mathcal{N}$ of subsets of X is a *network* for X if it is a network modulo $\{\{x\}\colon x \in X\}$. The *network weight* of X, $\mathrm{nw}(X)$, is the first cardinal number κ such that X has a network of cardinality κ.

For a space X, we denote by $C_p(X)$ the set of all real-valued continuous functions having X as its domain, with its pointwise convergence topology.

From now on, we will fix a countable base $\mathcal{B}(\mathbb{R})$ for the usual topology in the set of real numbers $\mathbb{R}$.

For subsets $E_1, \ldots, E_n$ of a space X and subsets $U_1, \ldots, U_n$ of $\mathbb{R}$, we will use the symbol $[E_1, \ldots, E_n; U_1, \ldots, U_n]$ to denote the set $\{f \in C_p(X)\colon f(E_i) \subset U_i \text{ for } i = 1, \ldots, n\}$. If $\mathcal{E}$ is a family of subsets of X, then $\mathcal{W}(\mathcal{E})$ will be the family of all the sets of the form $[E_1, \ldots, E_n; B_1, \ldots, B_n]$ where $E_1, \ldots, E_n \in \mathcal{E}$, $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ and $n \in \omega$.

Let Y be a subspace of X. By π_Y we denote the function from $C_p(X)$ to $C_p(Y)$ which restricts each element in $C_p(X)$ to Y; that is, $\pi_Y(f) = f \restriction Y$. For $y \in X$ we denote $\pi_{\{y\}}$ simply as π_y.

If $X = \prod\{X_t\colon t \in T\}$ is a topological product, $t \in T$ and $E \subset T$, then p_t and p_E denote the natural projections onto X_t and $\prod\{X_t\colon t \in E\}$, respectively.

Recall that given a space X, $\mathrm{iw}(X)$ represents the minimum weight of all possible Tychonoff topologies that are weaker than the topology of X.