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Maximal pairwise complementary families of quasi-uniformities

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For any positive integer $n > 1$ we construct on an infinite set a maximal pairwise complementary family of partial orders that has n elements. The example is motivated by a question of J. Steprāns and S. Watson.

1. Introduction

We consider a special case of Problem 3 of [11], which reads as follows:

What are the possible cardinalities of maximal families of mutually complementary families of partial orders?

When asking the question the authors of [11] state that they know of no finite maximal mutually complementary families of partial orders of cardinality greater than two on an infinite set. In this note we show that any finite positive integer number $n > 1$ can be the cardinality of such a family.

More generally in this paper our result is interpreted as a statement about the possible cardinalities of maximal mutually complementary families of T_0-quasi-uniformities. In this setting quasi-uniformities can indeed be understood as filtered generalizations of preorders.

Let us note that in [1] Anderson studied the number of mutually complementary preorders on a set of finite cardinality n.

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Furthermore in [2] Brown and Watson investigated the number of mutually complementary partial orders on a set of finite cardinality n. Among other things they showed [2, Theorem 3] that the number of mutually complementary partial orders on a set of cardinality n is at most $0.486n$ for all sufficiently large n.

On the other hand Steprāns and Watson showed in [11, Theorem 7] that if κ is an infinite cardinal and $3 \leqslant \nu \leqslant \kappa$, then there is a maximal family of ν many mutually complementary equivalence relations on κ. In fact these families are maximal even among preorders.

2. The result

We need some fundamental facts about binary relations defined on a set X. We shall call a reflexive transitive relation on a set X a *preorder*. A preorder that is antisymmetric will be called a *partial order*. For binary relations A, B on X we set $B \circ A = \{(x, z) \in X \times X\colon$ there is $y \in X$ such that $(x, y) \in A$ and $(y, z) \in B\}$.

A filter $\mathcal{U}$ on $X \times X$ such that each $U \in \mathcal{U}$ is a reflexive relation and for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \circ V \subseteq U$ is called a *quasi-uniformity* on X. Note that for any quasi-uniformity $\mathcal{U}$ the filter $\mathcal{U}^{-1} = \{U^{-1}\colon U \in \mathcal{U}\}$ where $U^{-1} = \{(y, x) \in X \times X\colon (x, y) \in U\}$ denotes the relation inverse to U, is also a quasi-uniformity on X. A quasi-uniformity $\mathcal{U}$ satisfying $\mathcal{U} = \mathcal{U}^{-1}$ is called a *uniformity*. The topology induced by $\mathcal{U}$ on X consists of all subsets G of X such that for each $x \in G$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq G$ where $U(x) = \{y \in X\colon (x, y) \in U\}$.

It is well known that the set $q(X)$ of all quasi-uniformities on a given set X yields a complete lattice provided that it is ordered under set-theoretic inclusion $\subseteq$. This lattice was studied to some extent in [3–8]. The smallest element of this lattice is the indiscrete uniformity $\mathcal{I} = \{X \times X\}$, while the largest element of this lattice is the discrete uniformity $\mathcal{D}$ generated by the diagonal $\Delta = \{(x, x)\colon x \in X\}$ of X. The lattice of preorders on a set X embeds as a sublattice (see [3, p. 3153]) into the lattice of quasi-uniformities on X, via the embedding $P \mapsto \mathcal{U}_P$ where $\mathcal{U}_P$ is the quasi-uniformity on X having base $\{P\}$. Note that under this correspondence partial orders yield T_0-quasi-uniformities. (We recall that a quasi-uniformity $\mathcal{U}$ on X is called a T_0-*quasi-uniformity* provided that the preorder $\bigcap \mathcal{U}$ associated with the quasi-uniformity $\mathcal{U}$ is a partial order.)

On a finite ground set X any quasi-uniformity $\mathcal{U}$ is generated by the base $\{\bigcap \mathcal{U}\}$. Hence for a finite set X the lattice of preorders can be identified with the lattice of quasi-uniformities.

The usual concept of a complement from lattice theory leads to the following notion: Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on the set X are called *complementary* if $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$ and $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$. (Of course, here $\wedge$ and $\vee$ denote the lattice operations of $q(X)$.)

With the help of Zorn's Lemma it is easy to verify that each mutually complementary family of quasi-uniformities on a set X can be extended to a maximal (with respect to inclusion) family of this kind.

For some recent work about lattices of uniformities we refer the reader to [13,14]. Complements in lattices of preorders were also investigated in [12]. Many basic facts about quasi-uniformities can be found in [10].

Without proofs we next present three fairly simple ideas that can be used to construct a mutually pairwise complementary family of partial orders possessing 2 (resp. 3) elements on an infinite set. With the help of Lemma 1 below the reader should have no difficulties to verify that the constructed families have indeed the stated properties. Afterwards we shall present a variant of our method that works for an arbitrary positive integer $n \geqslant 2$.

Example 1. If $\leqslant$ is a linear order on a set X possessing at least two elements, then, obviously, the filters generated on $X \times X$ by the bases $\{\leqslant\}$ and $\{\geqslant\}$ are T_0-quasi-uniformities on X. The constructed 2-element family of T_0-quasi-uniformities on X is (mutually) complementary and maximal with this property.