Topology and its Applications 165 (2014) 69–83

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Local minimalities in paratopological groups

Fucai Lin [1]

School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, PR China

ARTICLE INFO

Article history:
Received 15 November 2012
Received in revised form 26 October 2013
Accepted 21 January 2014

Dedicated to professor Shouli Jiang on the occasion of his 70th birthday

MSC:
22A30
22B05
52A30
46A63
46H05

Keywords:
Minimal topological groups
Minimal paratopological groups
Locally T_i-minimal paratopological groups
UFSS groups
Bisequential
Locally essential

ABSTRACT

In this paper, we mainly discuss the cardinal invariants on some class of paratopological groups. For each $i \in \{0, 1, 2, 3, 3.5\}$, we define the class of locally T_i-minimal paratopological groups by the conditions that, for a T_i paratopological group (G, τ), there exists a τ-neighborhood U of the neutral element such that U fails to be a neighborhood of the neutral element in any T_i-semigroup topology on G which is strictly coarser than τ. We mainly prove that (1) each UFSS and T_i-paratopological Abelian group (G, τ) is locally T_i-minimal; (2) if (G, τ) is a regular locally T_1-minimal Abelian paratopological group then $\chi(G) = \pi\chi(G)$; (3) if (G, τ) is an Abelian locally T_3-minimal paratopological group then we have $w(G) = nw(G)$. Moreover, we also discuss some relations of locally T_i-minimal paratopological groups and some properties of subgroups of T_i-minimal paratopological groups. Some questions are posed.

1. Introduction

All spaces are T_0 unless stated otherwise. Moreover, we assume that T_3 and $T_{3.5}$ spaces are T_1. By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the set of real numbers, the set of rational numbers, the set of irrational numbers, the set of integers and the set of positive integers, respectively. The letter e denotes the neutral element of a group. For a topological space (X, τ), a set $U \subset X$ is said to be a *τ-neighborhood* of some point $b \in X$ if it is open in (X, τ) and $b \in U$ (not to confuse, we say that U is a neighborhood of b). Readers may refer to [2,16,17] for notation and terminology not explicitly given here.

A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ into G is jointly continuous. If G is a paratopological group and the inverse map of G onto itself associating x^{-1} with

E-mail address: linfucai2008@aliyun.com.

[1] Supported by the NSFC (No. 11201414), the Natural Science Foundation of Fujian Province (No. 2012J05013) of China and Training Programme Foundation for Excellent Youth Researching Talents of Fujian's Universities (JA13190).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2014.01.018

arbitrary $x \in G$ is continuous, then G is called a *topological group*. However, there exists a paratopological group which is not a topological group; Sorgenfrey line [16, Example 1.2.2] is such an example.

Minimal topological groups were introduced independently by Doïtchinov [13] and Stephenson [29]: a Hausdorff topological group (G,τ) is called *minimal* if there is no Hausdorff group topology on G which is strictly coarser than τ. The readers can see recent advances in this field in [11]. The notion of locally minimal topological groups was introduced by Morris and Pestov in [24]: a Hausdorff topological group (G,τ) is called *locally minimal* if there exists a τ-neighborhood U of the neutral element such that U fails to be a neighborhood of the neutral element in any Hausdorff topology on G which is strictly coarser than τ. Recently, L. Außenhofer, T. Banakh, M.J. Chasco, D. Dikranjan and X. Domínguez have done much research in the locally minimal topological groups, see [4–6]. In the past few years, the study of paratopological groups has become an interesting topic in topological algebra, see [1–3,7–9,19–23]. Similarly to the case of topological groups, I. Guran defines the concept of minimal Hausdorff paratopological groups in [18]. Therefore, we can define the concept of local minimality in the class of paratopological groups. However, we have to specify a separation axiom in the case of paratopological groups since, unlike in topological groups, no implication $T_i \Rightarrow T_j$ with $0 \leqslant i < j \leqslant 3$ is valid in the class of paratopological groups. Moreover, it is an old open problem whether $T_3 \Rightarrow T_{3.5}$ in the class of paratopological groups. Therefore, for each $i \in \{0,1,2,3,3.5\}$, we define the class of locally T_i-minimal paratopological groups by the conditions that, for a T_i-paratopological group (G,τ), there exists a τ-neighborhood U of the neutral element such that U fails to be a neighborhood of the neutral element in any T_i-semigroup topology on G which is strictly coarser than τ.

This paper is organized as follows. In Section 3 we define the T_i-minimal paratopological groups and discuss the relations of T_i-minimal paratopological groups and minimal topological groups, where $i \in \{0,1,2,3,3.5\}$. In Section 4 we define the locally T_i-minimal paratopological groups, and discuss the relations of locally T_i-minimal paratopological groups and locally minimal topological groups, where $i \in \{0,1,2,3,3.5\}$. The aim of Section 5 is to discuss the cardinal invariants in locally T_i-minimal paratopological groups, where $i \in \{0,1,2,3,3.5\}$. We show that if (G,τ) is a regular locally T_1-minimal Abelian paratopological group then $\chi(G) = \pi\chi(G)$. In particular, a regular, bisequential locally T_1-minimal Abelian paratopological group is first-countable. Moreover, we also show that if (G,τ) is an Abelian locally T_3-minimal paratopological group then we have $w(G) = nw(G)$. In Section 6 we present some results about subgroups of T_i-minimal paratopological groups, where $i \in \{0,1,2,3,3.5\}$.

2. Preliminaries

Recall that a family $\mathcal{U}$ of non-empty open sets of a space X is called a *π-base* if for each non-empty open set V of X, there exists a $U \in \mathcal{U}$ such that $V \subseteq U$. The *π-character* of x in X is defined by $\pi\chi(x,X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a local π-base at x in $X\}$. The *π-character of* X is defined by $\pi\chi(X) = \sup\{\pi\chi(x,X)\colon x \in X\}$.

The weight of a topological space X is the minimal cardinality of a basis for its topology; it will be denoted by $w(X)$. The netweight of X is the minimal cardinality of a network in X, that is, a family $\mathscr{O}$ of subsets of X such that for any $x \in X$ and any open set U containing x there is $O \in \mathscr{O}$ with $x \in O \subseteq U$. The netweight of a space X will be denoted by $nw(X)$. The pseudocharacter $\psi(X,x)$ of a space X at a point x is the minimal cardinality of a family of neighborhoods of x whose intersection is $\{x\}$. The Lindelöf number $l(X)$ of a space X is the minimal cardinal κ such that any open cover of X admits a subcover of cardinality not greater than κ.

Recall that the topology of a space X is *determined* by a family $\mathscr{C}$ of its subsets provided that a set $F \subseteq X$ is closed in X if and only if $F \cap C$ is closed in C, for each $C \in \mathscr{C}$. If the topology of X is determined by a family of countably many compact subsets, then X is called a *k_ω-space*.