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Some new classes of topological spaces and annihilator ideals

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By a characterization of semiprime *SA*-rings by Birkenmeier, Ghirati and Taherifar in [4, Theorem 4.4], and by the topological characterization of $C(X)$ as a Baer-ring by Stone and Nakano in [11, Theorem 3.25], it is easy to see that $C(X)$ is an *SA*-ring (resp., *IN*-ring) if and only if X is an extremally disconnected space. This result motivates the following questions: Question (1): What is X if for any two ideals I and J of $C(X)$ which are generated by two subsets of idempotents, $Ann(I) + Ann(J) = Ann(I \cap J)$? Question (2): When does for any ideal I of $C(X)$ exists a subset S of idempotents such that $Ann(I) = Ann(S)$? Along the line of answering these questions we introduce two classes of topological spaces. We call X an *EF* (resp., *EZ*)*-space* if disjoint unions of clopen sets are completely separated (resp., every regular closed subset is the closure of a union of clopen subsets). Topological properties of *EF* (resp., *EZ*)*-spaces* are investigated. As a consequence, a completely regular Hausdorff space X is an F_α-space in the sense of Comfort and Negrepontis for each infinite cardinal α if and only if X is an *EF* and *EZ*-space. Among other things, for a reduced ring R (resp., $J(R) = 0$) we show that $Spec(R)$ (resp., $Max(R)$) is an *EZ*-space if and only if for every ideal I of R there exists a subset S of idempotents of R such that $Ann(I) = Ann(S)$.

1. Preliminaries

A space X is *extremally disconnected* (resp., *basically disconnected*) if the closure of every open subset is clopen in X (resp., if the closure of any cozeroset is open). It is well known that X is an extremally disconnected space if and only if any two disjoint open subsets of X are completely separated if and only if every open subset of X is C^*-embedded. In an extremally disconnected space all dense subsets are C^*-embedded. The reader is referred to [7, 1.H], [8,17]. A topological space is said to be *zero-dimensional* if it is a non-empty T_1-space with a base consisting of clopen sets. Zero-dimensional spaces were defined by Sierpiński in [15]. All zero-dimensional spaces are completely regular. A zero-dimensional space need not be a normal space. The space $\beta T = W^* \times N^*$ where T is the Tychonoff plank is an example of non-normal zero-dimensional

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space (see [7, Example 16.18]). A space X is *totally disconnected* if and only if the components in X are the singletons. Equivalently, X is totally disconnected if and only if the only non-empty connected subsets of X are the one-point sets. The following implications characterize the relationship among the notions defined above: X is extremally disconnected and $T_3 \Rightarrow X$ is zero-dimensional $\Rightarrow X$ is totally disconnected. None of the implications can be reversed and counterexamples exist even in the class of metric spaces. For terminology and notations, the reader is referred to [6] and [7].

All rings are assumed to be commutative with identity. By a reduced ring, we mean a ring without nonzero nilpotent elements. For each subset S of a ring R, $Ann(S) = \{r \in R\colon\ rs = 0,\ \forall s \in S\}$. A ring R is called an *SA* (resp., *IN*)-*ring* if for any two ideals I, J of R, $Ann(I) + Ann(J) = Ann(K)$, for some ideal K of R (resp., $Ann(I) + Ann(J) = Ann(I \cap J)$) (see [4]). We denote the Jacobson radical of R by $J(R)$. For terminology and notations, the reader is referred to [9].

Throughout this paper, we denote by $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff space X, and $C^*(X)$ is its subring of bounded functions. A completely regular Hausdorff space X is an *F-space* if its cozerosets are C^*-embedded. Equivalently, X is an F-space if finitely generated ideals of $C(X)$ are principal. For terminology and notations, the reader is referred to [7].

For the proof of the following lemma see [6, Corollary 3.6.5].

Lemma 1.1. *If A is a clopen subset of a topological space X, then $cl_{\beta X} A$ is a clopen subset of βX.*

For the proof of the following theorems see [7, 1.17] and [7, 1.15].

Theorem 1.2. *A subset S of X is C^*-embedded in X if and only if any two completely separated sets in S are completely separated in X.*

Theorem 1.3. *Two sets are completely separated if and only if they are contained in disjoint zero-sets. Moreover, completely separated sets have disjoint zero-set-neighborhoods.*

2. *EF*-space

We call a topological space X an *EF-space* if for any two collections $\mathcal{U}$ and $\mathcal{V}$ of clopen subsets of X with $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$, we have $\bigcup \mathcal{U}$ and $\bigcup \mathcal{V}$ are completely separated. The class of *EF*-space contains the class of spaces which are sums of connected spaces, all spaces for which the closure of any union of clopen subsets is open (hence all extremally disconnected spaces) and all spaces which any union of clopen subsets is C^*-embedded. If we take X as the sum of $\mathbb{R}$ (with usual topology) and $\mathbb{N}$, then X is an *EF*-space which is neither connected nor extremally disconnected. A zero-dimensional space need not be an *EF*-space. In fact, it is straightforward to check that if X is a zero-dimensional space, then X is an *EF*-space if and only if it is extremally disconnected. In this section, we prove that for any two ideals I and J of $C(X)$, which are generated by two subsets of idempotents, $Ann(I) + Ann(J) = Ann(I \cap J)$ (Question 1) if and only if X is an *EF*-space if and only if βX is an *EF*-space.

Lemma 2.1. *Let X be a normal space. The following statements are equivalent.*

(a) *X is an EF-space.*
(b) *For any two collections $\mathcal{U}$ and $\mathcal{V}$ of clopen subsets of X with $\bigcup \mathcal{U} \cap \bigcup \mathcal{V} = \emptyset$, we have $cl(\bigcup \mathcal{U}) \cap cl(\bigcup \mathcal{V}) = \emptyset$.*

Proof. (a) $\Leftrightarrow$ (b) This follows from the definition of *EF*-space and the fact that in a normal space disjoint closed subsets are completely separated. □