Topology and its Applications 165 (2014) 121–132

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Positive expansive flows

Alfonso Artigue

Departamento de Matemática y Estadística del Litoral, Universidad de la República, Gral. Rivera 1350, Salto, Uruguay

A R T I C L E   I N F O

Article history:
Received 22 July 2013
Received in revised form 23 January 2014
Accepted 23 January 2014

Dedicated to the memory of Guillermo Artigue

Keywords:
Expansive flows
Dynamical systems

A B S T R A C T

We show that every positive expansive flow on a compact metric space consists of a finite number of periodic orbits and fixed points.

0. Introduction

In the subject of discrete-time dynamical systems, a homeomorphism $f: X \to X$ on a compact metric space (X, d) is said to be *positive expansive* if there is $\alpha > 0$ such that if $\mathrm{d}(f^n(x), f^n(y)) < \alpha$ for all $n \geqslant 0$ then $x = y$. It is well known that if X admits a positive expansive homeomorphism then X is finite. The first proof of this result can be found in [14] (see also [6,3] for other proofs). In this note we show the corresponding result for flows, giving an affirmative answer to Problem 5.6 in [8]. Recall that a continuous flow $\Phi: \mathbb{R} \times X \to X$ on a compact metric space X is said to be *positive expansive* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathrm{d}(\Phi_t(x), \Phi_{h(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ and some increasing homeomorphism $h: \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ then $y = \Phi_s(x)$ with $|s| < \varepsilon$. We prove that if X admits a positive expansive flow then X is a finite union of circles and isolated points.

No proof in the discrete case can be directly adapted to the case of flows because expansiveness of flows is defined using reparameterizations of time. So, new techniques are needed. In Section 1 we introduce a new metric, equivalent to the given one, that has regular properties in relation with the flow (see Proposition 1.7). This metric does not depend on the expansiveness of the flow and seems to be of interest on its own. It allows us to prove that every point of a positive expansive flow is negative Lyapunov stable (allowing a time reparameterization). Having proved that, we find again that the proofs in the discrete case cannot be adapted. For example, [6] considers a finite covering $U_1, \ldots, U_n$ such that $\operatorname{diam} f^{-k}(U_i) \to 0$ as $k \to \infty$

E-mail address: artigue@unorte.edu.uy.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2014.01.015

and easily it is concluded that the space is finite. In the continuous case one has to reparameterize the trajectories, so this argument is not easy to adapt for flows. The diameters of open sets never decrease to zero with a flow without singular points. And if one introduces reparameterizations, the flowed open sets may not cover. So, some care is needed to conclude the proof. A different argument is given in Section 4.

Let us remark some facts about *translating* results from discrete dynamics to flows. If $f : X \to X$ is a homeomorphism then stable sets are usually defined as

$$W^s_\gamma(x) = \bigl\{y \in X\colon\ \mathrm{d}\bigl(f^n(x), f^n(y)\bigr) \leqslant \gamma \text{ for all } n \geqslant 0\bigr\}.$$

In the discrete-time case it is easy to prove that stable sets are closed sets. Consider $\Phi : \mathbb{R} \times X \to X$ a continuous flow on a compact metric space. Since the definition of expansiveness for flows considers reparameterizations it is natural to define stable sets allowing reparameterizations: $y \in W^s_\gamma(x)$ if there is a reparameterization $h : \mathbb{R} \to \mathbb{R}$ (increasing homeomorphism with $h(0) = 0$) such that

$$\mathrm{d}\bigl(\Phi_{h(t)}(y), \Phi_t(x)\bigr) \leqslant \gamma$$

for all $t \geqslant 0$. For flows, it is not true that $W^s_\gamma(x)$ is a closed set, see Example 9 in [17]. It seems to be a strange feature of the metric, so, it is natural to look for a new distance function (defining the same topology) with regular properties in relation with the flow (see Section 1).

In [10] it is proved that if X is a compact metric space admitting a minimal expansive homeomorphism then $\dim_{top}(X) = 0$. In the case of flows one would expect to conclude that $\dim_{top}(X) = 1$ if X admits an expansive minimal flow. In [7] techniques of local cross sections are used to attack the problem, but it remains an open case in order to give a complete translation (see Theorem 3.6 in [7]). This case is associated with *spiral orbits*. A point x is *spiral* if there is $t > 0$ such that $\Phi_t(x) \in W^s_\gamma(x) \cap H(x)$ where $H(x)$ is a small local cross section of the flow at x. In the discrete case, spiral points give rise to periodic orbits, but in the case of flows it seems to be still an open problem.

In [15] (the last line of the first page) one finds the following: "Having first found theorems in the diffeomorphism case, it is usually a secondary task to translate the results back into the differential equations framework". In the cited work, hyperbolic dynamics are considered. A special feature of hyperbolic flows is that stable sets need no reparameterizations. That is, if Φ is a hyperbolic flow then:

1. for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathrm{d}(\Phi_{h(t)}(x), \Phi_t(y)) < \delta$ for all $t \geqslant 0$ and some reparameterization h then $\mathrm{d}(\Phi_t(x), \Phi_t(y)) < \varepsilon$ for all $t \geqslant 0$ and moreover,
2. for small s it holds that $\mathrm{d}(\Phi_{t+s}(x), \Phi_t(y)) \to 0$ as $t \to \infty$.

It is a consequence of the Stable Manifold Theorem for hyperbolic flows. Therefore, techniques of diffeomorphisms can be adapted to flows, or at least they will not have to deal with reparameterizations. In light of this remark we wish to state the following problem. Given an expansive flow Φ, find a topologically equivalent one Φ' (i.e., both flows in X have the same orbits with the same orientation) satisfying items 1 and 2 above. The flow Φ' can be considered as a global reparameterization of Φ. If one is able to find such global reparameterization, stable sets would not need reparameterization and translations from expansive homeomorphisms to expansive flows would be easier. This seems to be an open problem. Another open problem is to define and construct a hyperbolic distance for an expansive flow as is done in [4] in the case of homeomorphisms.

Let us now describe the contents of this note. In Section 1 we define the Hausdorff metric for a flow and prove its main properties. In Section 2 some technical remarks are given related to expansiveness and reparameterizations. In Section 3 we show that the trajectories of a positive expansive flow are negative Lyapunov stable. In Section 4 we prove our main result and in Section 5 it is extended to positive expansiveness in the sense of Komuro [9] and singular points are allowed.