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Geometric automorphism groups of symplectic 4-manifolds

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ABSTRACT

Let M be a closed, oriented, smooth 4-manifold with intersection form $\varGamma$, $A(\varGamma)$ the automorphism group of $\varGamma$ and $D(M)$ the subgroup induced by orientation-preserving diffeomorphisms of M. In this note we study the question when $D(M)$ is of infinite index in $A(\varGamma)$ for a symplectic 4-manifold.

1. Introduction

For a given unimodular symmetric bilinear form $\varGamma$, let $A(\varGamma)$ be its automorphism group. Given a closed, oriented, topological 4-manifold M, let $\varLambda_M$ be the free abelian group obtained from $H^2(M;\mathbb{Z})$ by modulo torsion, and $\varGamma_M$ the associated unimodular symmetric bilinear form, namely, the intersection form on $\varLambda_M$. By a celebrated result of Freedman, any unimodular symmetric bilinear form is realized as the intersection form of an oriented, simply connected topological 4-manifold. Moreover, for such a topological manifold M, the natural map from the group of orientation-preserving homeomorphisms to $A(\varGamma_M)$ is surjective.

For a smooth, closed, oriented 4-manifold M with intersection form $\varGamma$, there is a natural map from the group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(M)$ to the automorphism group of $\varGamma$, $A(\varGamma)$. Let $D(M)$ be the image of this natural map. In other words, an automorphism is in $D(M)$ if it is realized by an orientation-preserving diffeomorphism. $D(M)$ is called the geometric automorphism group. The group $D(M)$, both as an abstract group and as a subgroup of $A(\varGamma)$, is a powerful smooth invariant, which is nonetheless hard to compute in general.

Wall initiated the comparison of $D(M)$ and $A(\varGamma)$ in a series of papers [19–21]. In particular, he proved in [21] the following beautiful result: for any simply connected smooth manifold with $\varGamma$ strongly indefinite or of rank at most 10, if there is an $S^2 \times S^2$ summand in its connected sum decomposition, then $D(M) = A(\varGamma)$.

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For Kähler surfaces, especially elliptic surfaces, rational surfaces, ruled surfaces, we have a rather good understanding of $D(M)$ due to Friedman, Morgan, Donaldson, Lönne [3,5,2,14] (see also [9,10]).

In this note we will focus on the question when $D(M)$ is of infinite index in $A(\Gamma_M)$ if $A(\Gamma_M)$ is an infinite group. We first observe in Theorem 2.3 that $A(\Gamma)$ is infinite if Γ is indefinite of rank at least 3. Moreover, we offer a simple criterion for a subgroup to have infinite index. We apply this criterion to symplectic manifolds and obtain an almost complete answer.

To state our result, let us first recall some definitions. For a smooth 4-manifold M with a symplectic form ω, let K_ω denote the symplectic canonical class. A symplectic 4-manifold is said to be minimal if it does not contain any embedded symplectic sphere with self-intersection -1. A general symplectic 4-manifold (M,ω) can be symplectically blown down to a minimal one, which is called a minimal model.

The Kodaira dimension of a symplectic 4-manifold (M,ω) is defined below.

Definition 1.1. If (M,ω) is minimal, the Kodaira dimension of (M,ω) is defined in the following way:

$$\kappa(M,\omega)=\begin{cases}-\infty & \text{if } K_\omega\cdot[\omega]<0 \text{ or } K_\omega\cdot K_\omega<0,\\ 0 & \text{if } K_\omega\cdot[\omega]=0 \text{ and } K_\omega\cdot K_\omega=0,\\ 1 & \text{if } K_\omega\cdot[\omega]>0 \text{ and } K_\omega\cdot K_\omega=0,\\ 2 & \text{if } K_\omega\cdot[\omega]>0 \text{ and } K_\omega\cdot K_\omega>0.\end{cases}$$

For a general (M,ω), $\kappa(M,\omega)$ is defined to be that of any of its minimal model.

It is shown in [7] that $\kappa(M,\omega)$ is well-defined and agrees with the holomorphic Kodaira dimension if (M,ω) is Kähler. Moreover, it turns out that $\kappa(M,\omega)$ only depends on M so we will denote it by $\kappa(M)$.

Theorem 1.2. *Suppose M has symplectic structures and $A(\Gamma_M)$ is infinite. Then $D(M)$ is of infinite index if*

- $\kappa(M)=-\infty$, *and* $M=\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ *with* $n\geqslant 10$ *or* $(\Sigma\times S^2)\#n\overline{\mathbb{CP}}^2$ *with* $n\geqslant 1$, *where* Σ *is a closed Riemann surface of positive genus.*
- $\kappa(M)=0$ *and* Γ_M *is odd.*
- $\kappa(M)\geqslant 1$.

This result follows from Propositions 3.2, 3.4, 3.3.

M is called a symplectic Calabi–Yau surface if there is a symplectic form ω on M such that K_ω vanishes in the real cohomology. The third author showed in [7] that M is a symplectic Calabi–Yau surface exactly when $\kappa(M)=0$ and Γ_M is even. With this understanding, Theorem 1.2 can be restated as: When M is symplectic and $A(\Gamma_M)$ is infinite, $D(M)$ is of finite index only when M is a symplectic Calabi–Yau surface, or $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ with $2\leqslant n\leqslant 9$.

We define Kähler Calabi–Yau surfaces in the same way. There are three Kähler Calabi–Yau surfaces with infinite $A(\Gamma)$: K3 surface, Enriques surface, T^4. All of them have finite index geometric automorphism group. The only known non-Kähler Calabi–Yau surfaces with infinite $A(\Gamma)$ are the so-called Kodaira–Thurston manifolds. We will show in the last section that they have infinite index geometric automorphism group. Thus we further make the following conjecture.

Conjecture 1.3. *Suppose M has symplectic structures and $A(\Gamma_M)$ is infinite. Then $D(M)$ is of finite index if and only if M is*

- *a Kähler Calabi–Yau surface, or*
- $\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ *with* $2\leqslant n\leqslant 9$.