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Non-weakly almost periodic recurrent points and distributionally scrambled sets on $\Sigma_2 \times \mathbb{S}^1$ ☆

Xinxing Wu [a,b,*], Guanrong Chen [b]

[a] School of Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, PR China

[b] Department of Electronic Engineering, City University of Hong Kong, Hong Kong Special Administrative Region

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ABSTRACT

Let $R_{r_0}, R_{r_1} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be irrational rotations and define $f : \Sigma_2 \times \mathbb{S}^1 \longrightarrow \Sigma_2 \times \mathbb{S}^1$ by

$$f(x,t) = \left(\sigma(x), R_{r_{x_1}}(t)\right),$$

for $x = x_1 x_2 \cdots \in \Sigma_2$, $t \in \mathbb{S}^1$, where $\Sigma_2 = \{0,1\}^{\mathbb{N}}$, $\mathbb{S}^1$ is the unit circle, $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ is a shift, and r_0 and r_1 are rotational angles. In this paper, it is proved that the system $(\Sigma_2 \times \mathbb{S}^1, f)$ has an uncountable distributionally ϵ-scrambled set for any $0 < \epsilon \leqslant \operatorname{diam} \Sigma_2 \times \mathbb{S}^1 = 1$ in which each point is recurrent but is not weakly almost periodic. This is a positive answer to a question posed in Wang et al. (2003) [6]. Furthermore, the following results are obtained:
(1) each distributionally scrambled set of f is not invariant;
(2) the system $(\Sigma_2 \times \mathbb{S}^1, f)$ is Li–Yorke sensitive.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{N} = \{1,2,3,\dots\}$ and $\mathbb{Z}^+ = \{0,1,2,\dots\}$. For a dynamical system (X,g) with metric d, the set of recurrent points, almost periodic points and weakly almost periodic points of g [8] are denoted by $\mathrm{R}(g)$, $\mathrm{A}(g)$ and $\mathrm{W}(g)$, respectively. Define the *positive orbit* of x by the set $\mathrm{orb}_g^+(x) = \{g^n(x)\colon n \in \mathbb{Z}^+\}$.

A point $x \in X$ is called *weakly almost periodic under g* if, for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|\{j\colon d(x, g^j(x)) < \epsilon,\ 0 \leqslant j < nN_\epsilon\}| \geqslant n$, where $|A|$ denotes the cardinality of set A.

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* Corresponding author at: School of Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, PR China.

E-mail addresses: wuxinxing5201314@163.com (X. Wu), gchen@ee.cityu.edu.hk (G. Chen).

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The notion of distributional chaos was first introduced in [4], where it was called 'strong chaos', which is characterised by a distributional function of distances between trajectories of two points. It is described as follows.

Let (X, g) be a dynamical system. For any pair $(x, y) \in X \times X$ and any $n \in \mathbb{N}$, the *distributional function* generated by g, x and y, $F^n_{x,y} : \mathbb{R} \longrightarrow [0, 1]$, is defined by

$$F^n_{x,y}(t, g) = \frac{1}{n}\left|\left\{i:\ d\left(g^i(x), g^i(y)\right) < t,\ 1 \leqslant i \leqslant n\right\}\right|.$$

Define the *lower and upper distributional functions* as

$$F_{x,y}(t, g) = \liminf_{n\to\infty} F^n_{x,y}(t, g),$$

and

$$F^*_{x,y}(t, g) = \limsup_{n\to\infty} F^n_{x,y}(t, g),$$

respectively. Both functions $F_{x,y}$ and $F^*_{x,y}$ are non-decreasing and $F_{x,y} \leqslant F^*_{x,y}$.

According to Schweizer and Smítal [4], a dynamical system (X, g) is *distributionally ϵ-chaotic* for some $\epsilon > 0$ if there exists an uncountable subset $\mathcal{S} \subset X$ such that for any pair of distinct points $x, y \in \mathcal{S}$, one has that $F^*_{x,y}(t, g) = 1$ for all $t > 0$ and $F_{x,y}(\epsilon, g) = 0$. The set $\mathcal{S}$ is called a *distributionally ϵ-scrambled set* and the pair (x, y) a *distributionally ϵ-chaotic pair*. If (X, g) is distributionally ϵ-chaotic for any $0 < \epsilon < \operatorname{diam} X$, then (X, g) is said to exhibit *maximal distributional chaos*. A pair $(x, y) \in X \times X$ is called a *distributionally chaotic pair* if it is a distributionally ϵ-chaotic pair for some $\epsilon > 0$. A set containing at least two distinct points is called a *distributionally scrambled set* if any pair of its distinct points is a distributionally chaotic pair. A dynamical system (X, g) is *distributionally chaotic*, if there exists an uncountable distributionally scrambled set in X.

Let $\Sigma = \{0, 1\}$, and consider a product space $\Sigma_2 = \Sigma^{\mathbb{N}}$ with the product topology. The space is compact and metrizable. Then, endow Σ_2 with the standard prefix metric

$$d_1(x, y) = \begin{cases} 0, & x = y, \\ \frac{1}{\min\{m \geqslant 1:\ x_m \neq y_m\}}, & x \neq y, \end{cases}$$

for any $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in \Sigma_2$.

Define $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ by $\sigma(x) = x_2x_3\cdots$ for any $x = x_1x_2\cdots \in \Sigma_2$, called the *shift* on Σ_2, which is continuous. Also, $(X, \sigma|_X)$ is called a *shift space* or *subshift*, where X is a closed and invariant subset of Σ_2.

Any element A of the set Σ^n is called an *n-word* over Σ and the *length* of A is n, denoted by $|A|$. A *word* over Σ is an element of the set $\bigcup_{n\in\mathbb{N}} \Sigma^n$. Let $A = a_1\cdots a_n \in \Sigma^n$ and $B = b_1\cdots b_m \in \Sigma^m$. Denote $AB = a_1\cdots a_nb_1\cdots b_m$ and $\overline{A} = \overline{a_1}\cdots\overline{a_n}$, where

$$\overline{a_i} = \begin{cases} 0, & a_i = 1, \\ 1, & a_i = 0. \end{cases}$$

Clearly $AB \in \Sigma^{n+m}$ and $\overline{A} \in \Sigma^n$. For any $a \in \Sigma$, denote a^n as an n-length permutation of a (for example, $0^3 = 000$), and $a^\infty = aa\cdots$ as an infinite permutation. If $x = x_1x_2\cdots \in \Sigma_2$ and $i \leqslant j \in \mathbb{N}$, then let $x_{[i,j]} = x_ix_{i+1}\cdots x_j$ and $x_{(i,j]} = x_{[i+1,j]}$. For any $B = b_1\cdots b_n \in \bigcup_{n\in\mathbb{N}} \Sigma^n$, the set $[B] = \{x_1x_2\cdots \in \Sigma_2:\ x_i = b_i,\ 1 \leqslant i \leqslant n\}$ is called the *cylinder* generated by B. For any $n \in \mathbb{N}$, let $\mathcal{B}_n = \{[b_1\cdots b_n]:\ b_i \in \Sigma,\ 1 \leqslant i \leqslant n\}$.