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Canonical embeddings of $S^1 \times \Delta^{n-1}$ into orientable n-dimensional closed PL manifolds for $n > 4$

D. Dimovski

Institute of Mathematics, Faculty of Natural Sciences and Mathematics, "Ss. Cyril and Methodius" University, Skopje, Macedonia

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ABSTRACT

For each embedded oriented circle c in an n-dimensional closed PL manifold M, we define a canonical embedding with respect to a triangulation of M, of $S^1 \times \Delta^{n-1}$ into a regular neighborhood of the embedded circle in M. For $n > 4$, we give a necessary and sufficient condition for an embedding of a surface with boundary in M, such that the embedding together with the canonical embeddings on the boundary components of the surface, extends to an embedding of the regular neighborhood of the surface in $\mathbb{R}^n$ to a regular neighborhood of the embedded surface in M.

0. Introduction

We will start with the following notation needed to state Theorem 0.0 which is a very well-known fact.

Let M be an n-dimensional, closed PL manifold, L its triangulation and K the first barycentric subdivision of L. Each $\sigma \in K^n$ can be represented as $[a_0 a_1 a_2 \cdots a_t \cdots a_{n-1} a_n]$, where a_i is the barycenter of an i-simplex $\sigma_i \in L^i$, such that $\sigma_0 < \sigma_1 < \cdots < \sigma_t < \cdots < \sigma_{n-1} < \sigma_n$ ($<$ means "a face of"). Let $\Delta^{n-1} = [012 \cdots n-1]$ and $\Delta^n = [012 \cdots n]$ be the standard $(n-1)$ and n-simplexes. Let SI be the segment $[-2,2]$ in $\mathbb{R}^1$ and let $W(SI) = [-1,1] \times \Delta^{n-1} \cup W(-2) \cup W(2)$ where $W(2) = 3 * (\Delta^{n-1} \times \{1\})$ and $W(-2) = (-3) * (\Delta^{n-1} \times \{-1\})$ are the cones on $\Delta^{n-1} \times \{1\}$ and $\Delta^{n-1} \times \{-1\}$ from 3 and -3 in $[-3,3] \times \Delta^{n-1}$. $(W(SI), W(-2), W(2))$ is a neighborhood triple of $(SI, -2, 2)$. We identify $W(-2)$ and $W(2)$ with Δ^n by identifying -3 and 3 with the vertex n from Δ^n, and by identifying $p \times \{-1\}$ and $p \times \{1\}$ with the vertex p of Δ^n, for each $p \neq n$. For an interior point X of an n-simplex $\sigma = [a_0 a_1 a_2 \cdots a_t \cdots a_{n-1} a_n]$ of K, let $X(a_j)$ be the middle point of the segment $[X, a_j]$ in σ, for each j. The span of all the points $X(a_j)$

E-mail address: donco@pmf.ukim.mk.

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is a neighborhood $N(X)$ of X in σ. Let $S^0 = \partial SI = \{-2, 2\}$, and let s be an embedding from S^0 into M, such that $s(-2) = A$ and $s(2) = B$ are interior points of two different n-simplexes of K. Using the above identification of $W(-2)$ and $W(2)$ with Δ^n we define the embedding $CE(s)$ from $W(-2) \cup W(2)$ into M by the homeomorphism which sends each vertex p of Δ^n to $A(a_p)$ for $W(-2)$ and to $B(a_p)$ for $W(2)$, and then extend linearly. Let $f : SI \to M$ be an embedding in general position with respect to K, and let its restriction on S^0 be denoted by s, such that $s(-2)$ and $s(2)$ are interior points of two different n-simplexes of K. We say that f is an arc in M and that s is its boundary. We denote by $\#f$ the number of intersection points of $f(SI)$ with the $(n-1)$ skeleton K^{n-1} of K.

Theorem 0.0. *Let f be an arc in M and let s be its boundary. Then, f together with $CE(s)$ extends to an embedding from $W(SI)$ into a neighborhood of $f(SI)$ in M if and only if $\#f$ is odd.*

The statement in Theorem 0.0, for $n = 0$ is vacuous.

Considering Theorem 0.0, as a zero-dimensional fact, in this paper we prove a one-dimensional analogy of it for orientable manifolds.

For an orientable n-dimensional manifold M, a regular neighborhood of an embedded circle in M is homeomorphic to $S^1 \times \Delta^{n-1}$. For each embedded oriented circle c in general position with respect to K, we define a canonical embedding, with respect to K, $CE(c)$ from $S^1 \times \Delta^{n-1}$ into a regular neighborhood of the embedded circle c in M. For each surface F with boundary we choose a standard surface SF in $\mathbb{R}^n$, and a regular neighborhood $W(SF)$ of SF in $\mathbb{R}^n$ such that its restriction on a boundary circle c of SF is a regular neighborhood $W(c)$ of the boundary circle c in $\mathbb{R}^n$. Let w be an embedding of SF in M in "good" general position with respect to K. We call such an embedding a surface in M. For each boundary circle c of SF, using the canonical embeddings, we define an embedding $\varphi(w(c))$ from $W(c)$ into a regular neighborhood of $w(c)$ in M. Let $\#w$ be the number if intersection points of $w(SF)$ with the $(n-2)$ skeleton K^{n-2} of K, and let $k(w)$ be the number of the boundary circles of the surface SF.

Theorem 0.1. *Let $n > 4$ and w be a surface in M. Then, w together with $\varphi(w(c))$ for the boundary circles c, extends to an orientation preserving embedding of $W(SF)$ in M whose image is a regular neighborhood of $w(SF)$ in M if and only if $\#(w) + k(w)$ is even.*

For orientable surfaces, Theorem 0.1 is Theorem 3.1, and the general case is Theorem 4.1.

In other words, in this paper we define a preferred framings on circles in orientable PL manifolds with respect to its triangulation, satisfying some good extra conditions. The canonical embeddings are related to the framed one-dimensional manifolds in $\mathbb{R}^n$ and the framed homology invariant of framed one-dimensional manifolds as discussed in [7]. The invariant in [7] is used to compute $n + 1$ and $n + 2$ homotopy groups of the n-dimensional sphere, but the question of canonical framings in general is not discussed.

As the fact Theorem 0.0 is used to characterize orientable manifolds, we will use the canonical embeddings and [2], to give several geometric description of spin manifolds and spin structures [5], such as the following two characterizations.

An orientable manifold M is spin if and only if for each embedded circle c it is possible to chose a preferred isotopy class of embeddings from $S^1 \times \Delta^{n-1}$ onto a neighborhood $N(c)$ of c in M, such that for any embedding of a surface with boundary in M, the embedding together with the preferred embeddings on the neighborhoods of the boundary circles extends to an embedding from the neighborhood of the surface in $\mathbb{R}^n$ to a neighborhood of its image in M.

A manifold M is spin if and only if each embedded one or two-dimensional closed manifold in M has a neighborhood homeomorphic to its neighborhood in $\mathbb{R}^n$.

The geometric characterization of spin manifolds is used to define indices for codimension one coincidences for maps on spin manifolds [3,4].