Topology and its Applications 160 (2013) 2196–2206

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Recurrence, pointwise almost periodicity and orbit closure relation for flows and foliations

Tomoo Yokoyama [1]

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido 060-0810, Japan

ARTICLE INFO

Article history:
Received 6 August 2012
Received in revised form 27 May 2013
Accepted 13 September 2013

Keywords:
Recurrence
Pointwise almost periodicity
Orbit closure relations
Flows
Foliations

ABSTRACT

In this paper, we obtain a characterization of the recurrence of a continuous vector field w of a closed connected surface M as follows. The following are equivalent: (1) w is pointwise recurrent. (2) w is pointwise almost periodic. (3) w is minimal or pointwise periodic. Moreover, if w is regular, then the following are equivalent: (1) w is pointwise recurrent. (2) w is minimal or the orbit space M/w is either $[0,1]$, or S^1. (3) R is closed (where $R := \{(x, y) \in M \times M \mid y \in \overline{O(x)}\}$ is the orbit closure relation). On the other hand, we show that the following are equivalent for a codimension one foliation $\mathcal{F}$ on a compact connected manifold: (1) $\mathcal{F}$ is pointwise almost periodic. (2) $\mathcal{F}$ is minimal or compact. (3) $\mathcal{F}$ is R-closed. Also we show that if a foliated space on a compact metrizable space is either minimal or both compact and without infinite holonomy, then it is R-closed.

1. Preliminaries

In [1] and [13], it is showed that the following properties are equivalent for a finitely generated group G on either a compact zero-dimensional space or a graph X: (1) (G, X) is pointwise recurrent. (2) (G, X) is pointwise almost periodic. (3) The orbit closure relation $R = \{(x, y) \in X \times X \mid y \in \overline{G(x)}\}$ is closed.

In this paper, we study the equivalence for these three notions for vector fields on surfaces and codimension one foliations on manifolds, and show the some equivalence. We assume that every quotient space has the usual quotient topology and that every decomposition consists of non empty elements. By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint subsets of a set X such that $X = \bigsqcup \mathcal{F}$. Let X be a topological space and $\mathcal{F}$ a decomposition of X. For any $x \in X$, denote by L_x the element of $\mathcal{F}$ containing x. Write $E_{\mathcal{F}} := \{(x, y) \mid y \in L_x\}$. Then $E_{\mathcal{F}}$ is an equivalence relation (i.e. a reflexive, symmetric and transitive relation). For a (binary) relation E on a set X (i.e. a subset of $X \times X$), let $E(x) := \{y \in X \mid (x, y) \in E\}$ for an element x of X. For any $A \subseteq X$, let $E(A) := \bigcup_{y \in A} E(y)$. A is said to be E-saturated if $A = \bigcup_{x \in A} E(x)$. Let $1_X := \{(x, x) \mid x \in X\}$ be the diagonal on $X \times X$. Thus $1_X \subseteq E$ if and only if E is reflexive (i.e. $x \in E(x)$

E-mail address: yokoyama@math.sci.hokudai.ac.jp.

[1] The author is partially supported by the JST CREST Program at Department of Mathematics, Hokkaido University.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.09.004

for all $x \in X$). Let $E^{-1} := \{(y, x) \mid (x, y) \in E\}$ (i.e. the image of E under the bijection $T : X \to X$ which interchanges coordinates). Clearly, E is symmetric if and only if $E = E^{-1}$. For any relation E on X, E is transitive if and only if $E(E(x)) \subseteq E(x)$. For an equivalence relation E, the collection of equivalence classes $\{E(x) \mid x \in X\}$ is a decomposition of X, denoted by $\mathcal{F}_E$. Note that decompositions (consisting of nonempty elements) can be corresponded to equivalence relations. Therefore we can identify decompositions with equivalence relations. For a relation E on a topological space X, E define the relation $\hat{E}$ on X with $\hat{E}(x) = \overline{E(x)}$. Denote by $\overline{E}$ the closure of E in $X \times X$. We call E pointwise almost periodic if $\hat{E}$ is an equivalence relation, R-closed if $\hat{E}$ is closed (i.e. $\hat{E} = \overline{E}$), compact if each element of E is compact, and it is minimal if each element of E is dense in X. By identification, we also said that $\mathcal{F}$ is R-closed if so is $E_{\mathcal{F}}$ and others are defined in similar ways. Notice that if $\mathcal{F}$ is either a foliation or the set of orbits of a flow, then $\hat{E}_{\mathcal{F}}$ is transitive. By a flow, we mean a continuous action of a topological group G on X. We call that $\mathcal{F}$ is trivial if it consists of singletons or is minimal. We characterize the transitivity for $\hat{E}$.

Lemma 1.1. *$\hat{E}$ is transitive if and only if $E(\hat{E}(x)) \subseteq \hat{E}(x)$. When $\hat{E}$ is an equivalence relation, this says that each $\hat{E}(x)$ is a union of E-equivalence classes.*

Proof. Since $\hat{E}(x)$ is closed, $E(y) \subseteq \hat{E}(x)$ implies $\hat{E}(y) = \overline{E(y)} \subseteq \hat{E}(x)$. □

Now we state a useful tool.

Lemma 1.2. *If E is an equivalence relation, then $\hat{E}$ is an equivalence relation if and only if it is symmetric (i.e. symmetry implies transitivity).*

Proof. Let $y \in \hat{E}(x)$ and $z \in E(y)$. By the symmetry assumption, we have $x \in \hat{E}(y)$. Since E is an equivalence relation, $E(y) = E(z)$ and so $\hat{E}(y) = \hat{E}(z)$. So $x \in \hat{E}(z)$ and, by symmetry again, $z \in \hat{E}(x)$ and so $E(y) \subseteq \hat{E}(x)$. Then $E(\hat{E}(x)) \subseteq \hat{E}(x)$. Hence $\hat{E}$ is transitive by Lemma 1.1. □

Notice that the twist map $T : X \to X$ is a homeomorphism and so $E = E^{-1}$ implies $\overline{E} = \overline{E}^{-1}$. In particular, $\overline{E}$ is symmetric whenever E is an equivalence relation. In particular, the previous lemma implies the following statement.

Corollary 1.3. *Suppose that E is an equivalence relation. If $\hat{E}$ is closed, then $\hat{E}$ is an equivalence relation.*

This is interpreted as the following statement.

Corollary 1.4. *If $\mathcal{F}$ is an R-closed decomposition, then $\mathcal{F}$ is pointwise almost periodic.*

The converse of this corollary is not true (see Example 4). A map $p : X \to Y$ is said to be perfect if it is continuous, closed, surjective and each fiber $p^{-1}(y)$ for any $y \in Y$ is compact. Recall that a net is a function from a directed set to a topological space.

Lemma 1.5. *If a relation E is closed and X is T_1, then $E(x)$ is closed for any $x \in X$. Moreover if E is an equivalence relation and X is compact Hausdorff, then the quotient map $q : X \to X/E$ is perfect.*

Proof. Note that each singleton is closed in a T_1-space. Since $\{x\} \times E(x) = E \cap (\{x\} \times X)$, we have $E(x)$ is closed in X. Suppose that E is an equivalence relation and X is compact Hausdorff. We will show that q is closed. Otherwise there is a closed subset B of X such that $E(B)$ is not closed. Fix any $y \in \overline{E(B)} - E(B)$. Let (y_α) be a net in B and $x_\alpha \in E(y_\alpha)$ such that $y_\alpha \to y$. Then $(x_\alpha, y_\alpha) \in E$. Since B is closed and so compact, we may assume that (x_α) converges to some element $x \in B$, by taking a subnet of (x_α). Since