Topology and its Applications 160 (2013) 1853–1861

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid

Kelly B. Yancey

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green St., Urbana, IL 61801, USA

ARTICLE INFO

Article history:
Received 7 January 2013
Received in revised form 24 June 2013
Accepted 18 July 2013

Keywords:
Topological weak mixing
Uniform rigidity
Klein bottle

ABSTRACT

In this paper we prove that there is a large family of topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid. We do this by viewing the Klein bottle as the quotient of the two-torus by an appropriate group action and producing topologically weakly mixing homeomorphisms of the two-torus that are uniformly rigid and equivariant with respect to the action.

1. Introduction

In ergodic theory, transformations that are of particular interest are ones that are rigid and also weakly mixing, see [1] for more information. An invertible measure-preserving map T defined on a Lebesgue space (X,μ) is called *rigid* if there exists an increasing sequence of natural numbers (n_m) such that $\mu(T^{n_m}A\Delta A)$ converges to zero for every measurable set A and is called *weakly mixing* if there exists a sequence (s_m) of density one such that $\mu(T^{s_m}A\cap B)$ converges to $\mu(A)\mu(B)$ for every A, B of positive μ-measure. Rigidity and weak mixing are two properties of a dynamical system that are very different, though not exclusive. Rigidity implies that at certain times the image of an interval is close to the interval, while weak mixing implies that at other times the images of intervals are evenly distributed. It is well known that weakly mixing, rigid maps are typical in the sense that they form a dense G_δ subset of all invertible measure-preserving transformations of a Lebesgue space (with respect to the weak topology) [7].

In this article we are interested in studying topological versions of weak mixing and rigidity. Let X be a compact metric space and $T: X\to X$ a homeomorphism. In this case, we call (X,T) a *flow*. A flow (X,T) is called *topologically transitive* if for every pair U, V of nonempty open subsets of X, there exists a time k with $T^kU\cap V\neq\emptyset$. This is sometimes referred to as *topological ergodicity* and is equivalent to the existence

E-mail address: kbyancey1@gmail.com.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.07.014

of a point with a dense orbit. A flow (X,T) is called *topologically weakly mixing* if $X \times X$ is topologically transitive.

Uniform rigidity in topological dynamics was introduced by Glasner and Maon in 1989 [4]. A flow (X,T) is *uniformly rigid* if there exists an increasing sequence of natural numbers (n_m) such that T^{n_m} converges to the Identity uniformly on X. This is a stronger notion of rigidity for compact metric spaces and is the topological analogue of rigidity in ergodic theory. As with rigidity, generic properties hold: there are large families of weakly mixing, uniformly rigid homeomorphisms of certain compact metric spaces. However, unlike rigidity, which does not depend on the space, the existence of weakly mixing, uniformly rigid homeomorphisms does. For example, in 2009 Silva et al. [8] showed that on a Cantor space there are no weakly mixing, uniformly rigid, measure-preserving maps with respect to any metric compatible with the topology (expect when the measure is concentrated at one point). In contrast, Yancey [9] produced a large family of such maps for the two-torus using techniques from Glasner and Weiss [5].

Let $\mathcal{Q}$ be the closure of the set of conjugations of an aperiodic rotation by Lebesgue measure-preserving homeomorphisms of $\mathbb{T}^2$. In [9] the following is proved:

Theorem 1.1. *There exists a dense G_δ subset $\mathcal{F}$ of $\mathcal{Q}$ such that for every $T \in \mathcal{F}$, $(\mathbb{T}^2, T, \mu)$ is weakly mixing, uniformly rigid, and strictly ergodic.*

While the author was presenting results from [9] in a seminar, Alica Miller asked if it would be possible to obtain similar generic results on the Klein bottle. This is an interesting question since constructing maps with specified topological properties has always presented difficulties. For example, the question of constructing a minimal homeomorphism of the Klein bottle presented mathematicians with trouble for some time, until 1965 when Robert Ellis produced such a map [3].

In Section 4 we produce a large family of topologically weakly mixing homeomorphisms that are uniformly rigid on the Klein bottle. Our approach is to view the Klein bottle as the quotient of $\mathbb{T}^2$ by an appropriate group action and produce homeomorphisms of $\mathbb{T}^2$ that are topologically weakly mixing, uniformly rigid, and equivariant with respect to the group action. Since these maps are equivariant and the desired properties are compatible with the projection to the Klein bottle, the maps induce homeomorphisms on the Klein bottle that are topologically weakly mixing and uniformly rigid.

Let $\mathcal{P}$ be the closure of the set of conjugations of an aperiodic rotation by homeomorphisms of the Klein Bottle. The main result of this paper is the following:

Theorem 1.2. *There exists a dense G_δ subset $\mathcal{S}$ of $\mathcal{P}$ such that every $T \in \mathcal{S}$ is topologically weakly mixing and uniformly rigid.*

In Section 2 we set notation and recall some definitions. Section 3 contains a description of the Klein bottle when viewed as the quotient of $\mathbb{T}^2$ by a group action and Section 4 contains a proof of Theorem 1.2. The final section of the paper, Section 5, discusses further questions related to uniform rigidity.

2. Background definitions

We will be considering homeomorphisms defined from X to X where X is a compact metric space with metric d. Since we will be discussing generic properties of these homeomorphisms we would like our space to be a complete metric space. Therefore we will be using the uniform distance between two homeomorphisms S, T of X given by

$$d_u(S,T) = \sup_{x \in X} d\big(S(x), T(x)\big) + \sup_{x \in X} d\big(S^{-1}(x), T^{-1}(x)\big).$$