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Factorization properties of paratopological groups

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ABSTRACT

In this article we continue the study of $\mathbb{R}$-factorizability in paratopological groups. It is shown that: (1) all concepts of $\mathbb{R}$-factorizability in paratopological groups coincide; (2) a Tychonoff paratopological group G is $\mathbb{R}$-factorizable if and only if it is totally ω-narrow and has property ω-QU; (3) every subgroup of a T_1 paratopological group G is $\mathbb{R}$-factorizable provided that the topological group G^* associated to G is a Lindelöf Σ-space, i.e., G is a *totally Lindelöf Σ-space*; (4) if $\Pi = \prod_{i\in I} G_i$ is a product of T_1 paratopological groups which are totally Lindelöf Σ-spaces, then each dense subgroup of Π is $\mathbb{R}$-factorizable. These results answer in the affirmative several questions posed earlier by M. Sanchis and M. Tkachenko and by S. Lin and L.-H. Xie.

1. Introduction

A *paratopological* (*semitopological*) group is a group with a topology such that multiplication on the group is jointly (separately) continuous. If in addition inversion on the group is continuous, then it is called a *topological* (*quasitopological*) group.

For every continuous real-valued function f on a compact topological group G, one can find a continuous homomorphism $p : G \to L$ onto a second-countable topological group L and a continuous real-valued function h on L such that $f = h \circ p$ (see [7, Example 37]). The conclusion remains valid for pseudocompact topological groups, a result due to W.W. Comfort and K.A. Ross [3]. These facts motivated the third listed author to introduce $\mathbb{R}$-*factorizable* groups in [15] as the topological groups G with the property that every continuous real-valued function on G can be factorized through a continuous homomorphism onto a second-countable topological group. The class of $\mathbb{R}$-factorizable groups is unexpectedly wide. For example,

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it contains arbitrary subgroups of σ-compact (and even Lindelöf Σ-) groups, topological products of Lindelöf Σ-groups, and their dense subgroups [14]. For other properties of this class of topological groups, the reader is referred to [16,17].

Similarly to the case of topological groups, M. Sanchis and M. Tkachenko introduced in [11] the classes of $\mathbb{R}_i$-factorizable paratopological groups, for $i \in \{1, 2, 3, 3.5\}$. The need in the use of four different subscripts was due to the fact that the classes of T_1, Hausdorff, regular and, possibly, completely regular paratopological groups are all distinct, while T_0 topological groups are completely regular. As in the case of topological groups, the classes of $\mathbb{R}_i$-factorizable paratopological groups are very wide. For example, it was proved in [11] that every Hausdorff (regular) Lindelöf totally ω-narrow paratopological group is $\mathbb{R}_2$-factorizable (resp., $\mathbb{R}_3$-factorizable), and that every subgroup of a Hausdorff (regular) σ-compact paratopological group is $\mathbb{R}_2$-factorizable (resp., $\mathbb{R}_3$-factorizable). Also, it was recently shown in [12] that for every continuous real-valued function f on a *feebly compact* paratopological group G (no separation requirement on G is imposed), one can find a continuous homomorphism $\pi : G \to H$ onto a compact metrizable *topological group* H and a continuous real-valued function h on H such that $f = h \circ \pi$. As usual, we call a space X feebly compact if every locally finite family of open sets in X is finite.

The following question was posed by M. Sanchis and M. Tkachenko in [11].

Question 1.1. ([11, Question 5.4]) Suppose that H is a Hausdorff paratopological group such that the associated topological group H^* is a Lindelöf Σ-space. Is every subgroup of H $\mathbb{R}_2$-factorizable?

We answer this question affirmatively in Theorem 4.8 even if H satisfies only the T_1 separation axiom.

For the further study of $\mathbb{R}$-factorizable topological groups, L.-H. Xie and S. Lin generalized the concept of uniform continuity of real-valued functions on topological groups. Modifying *property U* defined in [5], they introduced *property ω-U* and established that a topological group is $\mathbb{R}$-factorizable if and only if it is ω-narrow and has property ω-U (see [22, Theorem 4.9]). Recently, with the aim to study open homomorphic images of $\mathbb{R}_i$-factorizable paratopological groups, L.-H. Xie and S. Lin [23] extended property ω-U to paratopological groups. They proved that if G is a completely regular $\mathbb{R}_2$-factorizable ($\mathbb{R}_3$-factorizable) paratopological group and $p : G \to K$ is a continuous open homomorphism onto a paratopological group K satisfying $Hs(K) \leqslant \omega$ ($Ir(K) \leqslant \omega$), then K is $\mathbb{R}_2$-factorizable (resp., $\mathbb{R}_3$-factorizable). Here $Hs(G)$ and $Ir(G)$ stand, respectively, for the Hausdorff number and the index of regularity of the paratopological group G (see the definitions in [18]). We show in Proposition 3.20 that both restrictions $Hs(K) \leqslant \omega$ and $Ir(K) \leqslant \omega$ on the group K in [23] can be dropped.

It was also proved in [23] that every dense subgroup of a topological product of regular paratopological groups which are Lindelöf Σ-spaces is $\mathbb{R}_3$-factorizable and the following question was posed:

Question 1.2. ([23, Question 6.1]) Are dense subgroups of topological products of Hausdorff σ-compact paratopological groups $\mathbb{R}_2$-factorizable?

We give the positive answer to this question in Corollary 4.15 even in the case when the factors are T_1-spaces.

The article is organized as follows. In Section 3 we modify slightly the original definition of $\mathbb{R}_i$-factorizability given in [11] by eliminating the separation restrictions on a paratopological group G (but keeping these restrictions for second-countable continuous homomorphic images of G). Having a recourse to [21], we show that *all* variants of $\mathbb{R}$-factorizability in paratopological groups are equivalent to its weakest form, when the second-countable continuous homomorphic images of a given paratopological group are not assumed to satisfy any separation axiom.

Making use of property ω-QU (a form of property ω-U designed for paratopological groups), we characterize Tychonoff $\mathbb{R}$-factorizable paratopological groups. It is proved in Theorem 3.14 that every totally