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Special cycles in independence complexes and superfrustration in some lattices ☆

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ABSTRACT

We prove that the independence complexes of some grids have exponential Betti numbers. This corresponds to the number of ground states in the hard-core model in statistical physics with fermions in the vertices of the grid.

1. Introduction

The purpose of this paper is to investigate some topological questions arising from the study of supersymmetric lattice models in statistical physics. We will begin by explaining this motivation, but the mathematical content of the paper belongs to combinatorial algebraic topology.

Suppose one has a finite graph L, which in applications is usually a periodic lattice with some boundary conditions. Square, triangular or hexagonal grids are the most notable examples. The vertices of the graph can be occupied by particles, such as fermions, which satisfy the *hard-core* restriction: two adjacent vertices cannot be occupied simultaneously. A configuration of particles which satisfies this assumption is an *independent set* in the graph L.

Associated with L there is a simplicial complex called the *independence complex* of L and denoted $I(L)$. Its vertices are the vertices of L and its faces are the independent sets in L. It is a standard object studied in combinatorial algebraic topology. There is a close connection between the simplicial and topological invariants of $I(L)$ and certain characteristics of the corresponding lattice model which are of interest to physicists. It is beyond the scope of this paper to discuss this relationship in detail; we refer to [10] and we limit ourselves to presenting just the most basic dictionary:

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the partition function of L	the f-polynomial of $I(L)$,
the Witten index of L	minus the reduced Euler characteristic $-\widetilde{\chi}(I(L))$,
the number of zero energy ground states	the dimension of $\widetilde{H}_*(I(L);\mathbb{Q})$.

There has been some very successful work calculating the Witten index, homology groups or indeed the complete homotopy type of the independence complex for various lattices, e.g. [2,6–9,11–14,22]. In this paper we focus on the large-scale picture. Computer simulations of van Eerten [3] indicate that for some types of lattices, as their size increases, the number of ground states grows exponentially with the number of vertices, that is

$$\dim \widetilde{H}_*\big(I(L)\big) \sim a^{v(L)}$$

for some constant a depending on the type of the lattice, where $v(L)$ denotes the number of vertices in a graph L. This situation is called *superfrustration* and has interesting physical implications, see [9]. Engström [5] developed a general method of computing *upper* bounds for the constant a. For the lattices of [3] it gives bounds very close to the values predicted in [3].

This paper has two main parts. In the first one we present a method which can be used to construct exponentially many linearly independent homology classes in $I(L)$ for graphs L of certain type. That proves superfrustration of certain lattices and we give examples based on modifications of the triangular lattice. In the second part we prove a generalization of the main result of [5], which can sometimes give better upper bounds.

Both methods work particularly nicely with one type of lattice studied in [3,5]: the *hexagonal dimer*, also known as the *Kagome lattice* (see Fig. 1). Under suitable divisibility conditions on the height and width we will prove that a graph $\mathbb{H}$ of that type satisfies:

$$1.02^{v(\mathbb{H})} \approx \big(2^{1/36}\big)^{v(\mathbb{H})} \leqslant \dim \widetilde{H}_*\big(I(\mathbb{H})\big) \leqslant \big(14^{1/36}\cdot 2^{1/6}\big)^{v(\mathbb{H})} \approx 1.21^{v(\mathbb{H})}.$$

We will prove the lower bound in Section 4 and the upper bound in Section 6. The previous upper bound of [5] was $2^{1/3} \approx 1.26$ and the experimental approximation by [3] is 1.25 ± 0.1.

Our technique for lower bounds produces slightly more than just homology classes: we obtain a large wedge of spheres that splits off. For instance, for a suitable lattice $\mathbb{H}$ of Kagome type, this reads as a homotopy equivalence

$$I(\mathbb{H}) \simeq \left(\bigvee^{(2^{1/36})^{v(\mathbb{H})}} S^{2v(\mathbb{H})/9-1}\right) \vee X$$

for some space X. This type of result is proved in Section 5.

Remark 1.1. Estimations as above can be compared against the absolute upper bound: for any graph G we have

$$\dim \widetilde{H}_*\big(I(G)\big) \leqslant \big(2^{2/5}\big)^{v(G)} \approx 1.32^{v(G)}.$$

This follows from the results of [17]; for another short proof see [1].

2. Notation

Let $G = (V(G), E(G))$ be a simple, undirected graph. Our main object of interest is the independence complex of G.

Definition 2.1. An *independent set* in G is a subset $W \subseteq V(G)$ such that for any $u, v \in W$ we have $uv \notin E(G)$.

The *independence complex* $I(G)$ of the graph G is the simplicial complex with vertex set $V(G)$ whose faces are all the independent sets in G.

We define $v(G) = |V(G)|$. For any vertex v we write $N[v]$ for the *closed neighbourhood* of v, that is the set consisting of v and all its adjacent vertices in G. For any set $W \subseteq V(G)$ we define $N[W] = \bigcup_{v\in W} N[v]$.

For any simplicial complex K and a subset $U \subseteq V(K)$ of the vertices $K[U]$ denotes the induced subcomplex of K with vertex set U. The same notation is used for graphs. If H is an induced subgraph of G then $I(H)$ is an induced subcomplex of $I(G)$. By $|K|$ we denote the number of faces in K, including the empty one. The join $K * L$ of two complexes K and L with disjoint vertex sets is the simplicial complex whose faces are the unions $\sigma \cup \tau$ for $\sigma \in K$ and $\tau \in L$. For any two graphs G and H, if $G \sqcup H$ is their disjoint union, we have

$$I(G \sqcup H) = I(G) * I(H).$$

The symbol S^k denotes the topological sphere of dimension k. The (unreduced) suspension ΣK is the join $K * S^0$. If e is the graph consisting of a single edge then $I(e) = S^0$.

The reduced homology and cohomology groups of K, denoted $\widetilde{H}_*(K;\mathbb{Q})$, $\widetilde{H}^*(K;\mathbb{Q})$, are the homology groups of the augmented chain, respectively cochain complex of K. Throughout the paper we always use rational coefficients and omit them from notation. We have $\widetilde{H}_i(\Sigma K) = \widetilde{H}_{i-1}(K)$. There is a standard bilinear pairing, denoted $\langle\cdot,\cdot\rangle$:

$$\langle\cdot,\cdot\rangle : \widetilde{H}^i(K) \otimes \widetilde{H}_i(K) \to \mathbb{Q}$$