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Embedding suspensions into hyperspace suspensions

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ABSTRACT

We present sufficient conditions to guarantee that the topological suspension, $Sus(X)$, of a continuum X can be embedded in the hyperspace suspension, $HS(X)$, of X in such a way that the vertexes of $Sus(X)$ are sent to the two distinguished points of $HS(X)$. We characterize several classes of continua having this property.

1. Introduction

We present sufficient conditions to guarantee that the topological suspension, $Sus(X)$, of a continuum X can be embedded in the hyperspace suspension, $HS(X)$, of X in such a way that the vertexes of $Sus(X)$ are sent to the two distinguished points of $HS(X)$. We denote by $\mathcal{L}$ the class of continua whose topological suspension can be embedded in its hyperspace suspension in the way described above.

The paper is divided in six sections. After the definitions and notations, in Section 3, we give sufficient conditions for a continuum to belong to $\mathcal{L}$ (Theorem 3.1 and Example 3.2), and we characterize several classes of continua that belong to $\mathcal{L}$ (Examples 3.3, 3.4 and 3.5). In Section 4, we characterize pseudo-linear and pseudo-circular continua that belong to $\mathcal{L}$ (Theorem 4.3). We also characterize hereditarily decomposable C-H continua that belong to $\mathcal{L}$ (Theorem 4.4). In Section 5, we prove that smooth dendroids and fans belong to $\mathcal{L}$ (Theorems 5.3 and 5.5). In Section 6, we show that the arc and the simple closed curve are the only two decomposable atriodic continua in $\mathcal{L}$ (Corollary 6.6).

2. Definitions and notations

If (Z,d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the ε open neighborhood of A is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior, closure and boundary of A are denoted by $Int_Z(A)$, $Cl_Z(A)$, $Bd_Z(A)$, respectively. $Z \setminus A$ denotes the complement of A in Z and Z/A denotes the quotient space of Z modulo A with the quotient topology.

A *map* is a continuous function. Let Z be a metric space and let W be a subspace of Z. A map $r: Z \to W$ is a *retraction* provided that $r(w) = w$ for each $w \in W$.

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Let Z be a metric space. The *topological cone* of Z, denoted by $Cone(Z)$, is the quotient space $(Z \times [0,1])/(Z \times \{1\})$. The symbol v_Z denotes the point corresponding to $Z \times \{1\}$ in $Cone(Z)$ and it is called the *vertex of* $Cone(Z)$. Let $q_{CZ} : Z \times [0,1] \to Cone(Z)$ be the quotient map. Then given a point $z \in Z$, $q_{CZ}(\{z\} \times [0,1])$ is called the *coning arc over* z and it is denoted by α_z. The *base of the cone of* Z is the set $\mathcal{B}(Z) = q_{CZ}(Z \times \{0\})$. The *topological suspension* of Z, denoted by $Sus(Z)$, is the quotient space $(Z \times [-1,1])/\{Z \times \{-1\}, Z \times \{1\}\}$. The symbols v_Z^+ and v_Z^- denote the points in $Sus(Z)$ corresponding to $Z \times \{-1\}$ and $Z \times \{1\}$, respectively, and are called the *vertexes of* $Sus(Z)$. The quotient map from $Z \times [-1,1]$ onto $Sus(Z)$ is denoted by q_{SZ}.

A *continuum* is a compact, connected, metric space. A continuum X is *decomposable* provided that there exist two proper subcontinua K and L such that $X = K \cup L$. X is *indecomposable* if it is not decomposable. X is *hereditarily decomposable* provided that every nondegenerate subcontinuum of X is decomposable. X is *hereditarily indecomposable* if all its subcontinua are indecomposable.

A *graph* is a continuum which can be written as a finite union of arcs, any two of which are either disjoint or intersect at one of both of their end points. A *tree* is a graph that does not contain a simple closed curve.

A continuum X is *unicoherent* provided that if K and L are subcontinua of X such that $X = K \cup L$, then $K \cap L$ is connected. X is *hereditarily unicoherent* if all its subcontinua are unicoherent.

A continuum X is a *triod* if there exists a subcontinuum K of X such that $X \setminus K = M_1 \cup M_2 \cup M_3$, where each $M_j \neq \emptyset$ and $Cl_X(M_j) \cap M_k = \emptyset$ for $j \neq k$. A continuum X is *atriodic* if X does not contain a triod.

A subcontinuum Y of a continuum X is *terminal* provided that for each subcontinuum Z of X such that $Z \cap Y \neq \emptyset$ we have that either $Z \subset Y$ or $Y \subset Z$. A subcontinuum Y of the continuum X is an *end subcontinuum* if for any two subcontinua K and L of X with $Y \subset K \cap L$, we have that either $K \subset L$ or $L \subset K$.

The *Hilbert cube* is the countable product of copies of $[0,1]$, and it is denoted by $\mathcal{Q}$.

Given a continuum X we consider the following *hyperspaces*:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\};$$

$$\mathcal{C}(X) = \left\{A \in 2^X \;\middle|\; A \text{ is a subcontinuum of } X\right\};$$

$$\mathcal{F}_1(X) = \left\{\{x\} \;\middle|\; x \in X\right\}.$$

We topologize 2^X with the Hausdorff metric $\mathcal{H}$ [13, (0.1)]. It is known that 2^X and $\mathcal{C}(X)$ are continua [13, (1.13)]. Hence, we may consider the hyperspaces 2^{2^X} and $\mathcal{C}(2^X)$ topologized with the Hausdorff metric $\mathcal{H}^2$ induced by $\mathcal{H}$. Given $p \in X$, we let $\mathcal{C}_p(X) = \{K \in \mathcal{C}(X) \mid p \in K\}$. If $f : X \to Y$ is a map between continua, $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$ given by $\mathcal{C}(f)(A) = f(A)$ is the *induced map by* f [13, (0.49)]. An *order arc in* $\mathcal{C}(X)$ is a map $\gamma : [0,1] \to \mathcal{C}(X)$ such that if $s, t \in [0,1]$ and $s < t$, then $\gamma(s) \subsetneq \gamma(t)$.

A *Whitney map* is a map $\mu : \mathcal{C}(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(A) < \mu(B)$ for all $A, B \in \mathcal{C}(X)$ such that $A \subsetneq B$. If $t \in [0,1]$, then $\mu^{-1}(t)$ is called a *Whitney level*.

We also consider the following quotient space:

$$HS(X) = \mathcal{C}(X)/\mathcal{F}_1(X),$$

with the quotient topology. $HS(X)$ is called the *hyperspace suspension* of X and was originally defined in [14]. Let $q_X : \mathcal{C}(X) \to HS(X)$ be the quotient map. We denote by T_X the point $q_X(X)$ and by F_X the point corresponding to $q_X(\mathcal{F}_1(X))$.

Remark 2.1. Note that the sets $HS(X) \setminus \{F_X\}$ and $HS(X) \setminus \{T_X, F_X\}$ are homeomorphic to $\mathcal{C}(X) \setminus \mathcal{F}_1(X)$ and $\mathcal{C}(X) \setminus (\{X\} \cup \mathcal{F}_1(X))$, respectively, using the appropriate restriction of q_X.

Let $\mathcal{L}$ be the class of continua X which admit an embedding $\xi : Sus(X) \to HS(X)$ such that $\xi(\{v_X^+, v_X^-\}) = \{T_X, F_X\}$.

3. General results

We give sufficient conditions for a continuum to belong to $\mathcal{L}$ (Theorem 3.1 and Example 3.2). We characterize several classes of continua that belong to $\mathcal{L}$ (Examples 3.3, 3.4 and 3.5). We also present several examples of particular continua belonging to $\mathcal{L}$ and not belonging to $\mathcal{L}$.

Theorem 3.1. *A continuum X belongs to the class $\mathcal{L}$ in each of the following cases*:

(1) *There exists an embedding $i : Cone(X) \to \mathcal{C}(X)$ such that $i(\mathcal{B}(X)) \subset \mathcal{F}_1(X)$, $i(Cone(X) \setminus \mathcal{B}(X)) \subset \mathcal{C}(X) \setminus \mathcal{F}_1(X)$ and $i(v_X) = X$.*
(2) *There exists a homeomorphism $h : Cone(X) \to \mathcal{C}(X)$ with $h(v_X) = X$ and $h(\mathcal{B}(X)) = \mathcal{F}_1(X)$.*
(3) *There exists an embedding $\chi : Sus(X) \to \mathcal{C}_p(X)$ such that $\chi(\{v_X^+, v_X^-\}) = \{\{p\}, X\}$.*

Such an embedding χ exists, for example, in the following cases: