Topology and its Applications 160 (2013) 1155–1163

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The Semi-Polish Theorem: One-sided vs joint continuity in groups

A.J. Ostaszewski

Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom

A R T I C L E   I N F O

Article history:
Received 2 August 2012
Received in revised form 23 April 2013
Accepted 5 May 2013

MSC:
26A03
04A15
02K20

Keywords:
Automatic continuity
Analytic Baire Theorem
Analytic Cantor Theorem
Shift-compactness
Proper metric
Group-norm

A B S T R A C T

We give a symmetrized, unilaterally generated topological condition which ensures that a right-topological group is a Polish topological group.

1. Introduction

Let X be a group (with neutral element e) equipped with a right-invariant metric d_R^X, i.e. $d_R^X(xv, yv) = d_R^X(x, y)$, for all x, y, v in X. The metric generates a topology under which X is a *right-topological group* – the right-shifts $\rho_v(x): x \mapsto xv$ are continuous, and so homeomorphisms (in fact bi-uniformly continuous homeomorphisms as both ρ_v and $\rho_v^{-1} = \rho_{v^{-1}}$ are uniformly continuous). In any right-topological group such a compatible metric exists (see e.g. by [42, Th. 7.3.1]) iff there exists a metric for which the right-shifts are uniformly continuous. See Section 5.1 for a key example.

The conjugate left-topological group structure on X is obtained by taking $d_L^X(x, y) := d_R^X(x^{-1}, y^{-1})$. This is a left-invariant metric on X under which the left-shifts $\lambda_u(x): x \mapsto ux$ are bi-uniformly continuous.

Below we are concerned with the join of the two metric topologies (coarsest joint refinement), which is generated by the symmetrized metric

$$d_S^X := \max\{d_R^X, d_L^X\}.$$

Its general significance comes from the theorem that, for (T, d^T) any complete metric space, the group of bounded self-homeomorphisms of T is complete under the symmetrization of the supremum metric (cf. Section 5.1, where some additional results are derived; for details see [38] and [14, Th. XIV.2.6, p. 296]).

E-mail address: a.j.ostaszewski@lse.ac.uk.

0166-8641/$ – see front matter © 2013 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.topol.2013.05.006

Definitions. In the context of a group equipped with a right-invariant metric, for any topological property P, say that the group X has the *symmetrized-P property*, or more briefly: X is *semi-P*, if the d_S^X topology has property P. In particular, call the group X *semi-Polish* if it is symmetrized-Polish, so that (X, d_S^X) is topologically complete and separable. (This name was suggested by Anatole Beck.)

Recall that, in a Hausdorff space, a set is *analytic* if it is the continuous image of a Polish space (a separable metric space which is topologically complete) – see [20] for details. So say that the group X is *semi-analytic* if X is an analytic subset in the d_S^X topology.

Our main result is the following.

Theorem 1 *(Main Theorem: Semi-Polish Theorem). For a group X equipped with a right-invariant metric d_R^X: if the space X is non-meagre and semi-Polish (more generally, semi-analytic) under the topology of d_R^X, then it is a Polish topological group (i.e. under the d_R^X topology X is completely metrizable and a topological group).*

Of course every metrizable topological group has an equivalent right-invariant metric, by the celebrated Birkhoff–Kakutani Theorem ([4,24], cf. [38], viewed there as a 'normability' theorem), and a Polish group is non-meagre (Baire's Theorem), so this theorem covers all Polish groups.

The theorem also generalizes a result due to Loy [28] and Hoffmann-Jørgensen [22, Th. 2.3.6, p. 355] that a Baire analytic *topological* group is Polish, because an analytic group is separable and metrizable (for which see [22, Th. 2.3.6, p. 355]).

The theorem addresses the question: when does one-sided continuity of multiplication imply its joint continuity and further its *admissibility*, i.e. endowment of a topological group structure? That question was considered in the *abelian* context by Ellis in [15] (see in particular his Th. 2, where the topology is locally compact – cf. Section 5 below), but otherwise the existing literature, which goes back to Montgomery [32] and also Ellis [16] via Namioka [33], considers some form of weak bilateral continuity, usually separate continuity, supported by additional topological features, including some form of completeness. See Bouziad's two papers [5] and [6] for the state-of-the-art results, deducing automatic joint continuity from separate continuity (and for a review of the historic literature), and the more recent paper of Solecki and Srivastava [43], where separate continuity is weakened; cf. [10]. For the broader context of automatic continuity see [46] (e.g. p. 338), and for the interaction of topology and algebra, Dales [12].

By contrast to these bilateral conditions, the Main Theorem above assumes only a particular form of one-sided continuity, supported by additional topological properties. A contribution of this paper is to replace the use of local compactness (or even subcompactness, for which see [6]) by the recently isolated much weaker notion of shift-compactness in groups given here in the analytic format of Theorem IV in Section 4 (cf. [2]) and studied for its relationship to analyticity (definition below) in the companion paper [38], results required from there being identified in Section 4.

We are unaware of any similar analysis in the literature, save for the work of Itzkowitz and his collaborators: see e.g. [19] for a different analysis, conducted in the broader category of uniform spaces, which compares left and right uniformities. Our approach is guided here by the result of [2] (Th. 3.9 – Ambidextrous refinement) that (X, d_S^X) is a topological group iff (X, d_R^X) is a topological group.

We write $\|x\| := d_R^X(x, e)$; as $d_R^X(x, e) = d_R^X(e, x^{-1}) = d_L^X(x, e)$, it is appropriate to call this the associated *group-norm*. It conveniently describes the *right* and *left uniformities* generated by d_R^X, to which we refer below, via the respective relations $\|x^{-1}y\| < \varepsilon$ and $\|xy^{-1}\| < \varepsilon$. The right-invariant metric d_R^X is in fact retrievable from the norm in the sense given in Section 5.2 via $d_R^X(x, y) = \|x^{-1}y\|$. One also has $\|x\| = d_S^X(x, e)$, i.e. d_S^X also defines the same norm, which emphasizes that the symmetrized topology is imposed by the assumed one-sided structure of the right-topology.

We freely use the fact that an analytic set has the Baire property (cf. [25, Th. 21.6], the Lusin–Sierpiński Theorem, and the closely related Cor. 29.14, Nikodym Theorem, cf. the treatment in [26, Cor. 1, p. 482] or [20, pp. 42–43]; cf. [40]). We refer to this result as the *Lusin–Sierpiński–Nikodym Theorem*, abbreviated to LSN. Our interest in analyticity as carrier of the Baire property was motivated by van Mill's proof in [30] of an analytic form of the Effros Theorem, which de facto assumes only separate continuity of the group action on X and the existence of a right-invariant metric on X.

The rest of the paper is organized as follows. Section 2 is devoted to a proof of the Main Theorem and some related results on symmetrized properties are derived in Section 3; the arguments rely on some results obtained in the companion paper [38], so for self-sufficiency these are itemized in Section 4 as Theorems I–IV. The Complements Section (Section 5) identifies natural examples of groups with a right-invariant metric and comments on the standing of such structures to the Loy and Hoffmann-Jørgensen Theorems (above), and the Effros Open Mapping Principle; further generalizations are sought via a Vaĭnstein-like condition, and a non-separable variant of Theorem 1 is briefly discussed against the context of 'resolvable sets' (cf. [35]). The underlying theme is that these structures have the standing of a primitive notion as *normed groups*; we comment on why they are either topological groups or just "pathological".

Notation. We use the subscripts R, L, S as in $x_n \to_R x$ etc. to indicate convergence in the corresponding metrics d_R^X, d_L^X, d_S^X.