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The intersection properties of generalized Helly families for inverse limit spaces ☆

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The aim of this paper is to discuss the intersection properties of generalized Helly families for topological spaces and inverse limit spaces. This concept is a generalization of Helly family. A generalized Helly family $\mathcal{C}$ is a countable family of ∞-connected subsets of a topological space X satisfying the following conditions: the intersection $\bigcap \mathcal{E}$ of each finite subfamily $\mathcal{E} \subset \mathcal{C}$ is ∞-connected; and the intersection $\bigcap \mathcal{D}$ of each proper subfamily $\mathcal{D} \subset \mathcal{C}$ is nonempty.

In [6], Kulpa (1997) extended the Helly convex-set theorem onto topological spaces in terms of Helly families. Here, we improve his result. We show that if $\mathcal{C}$ is a generalized Helly family of compact subsets of a topological space X and $\mathcal{U}$ is a countable covering of X with $C_j \subset U_j$, for each $j \in \mathbb{N}$, then $\bigcap \mathcal{D}$ is nonempty.

1. Introduction

In the current article, we will introduce the concept of generalized Helly family. Then by using the tools of [6], we will provide some intersection properties for countable coverings of topological spaces and inverse limit spaces, which improve the results of [6].

The Helly theorem which was first published in 1921 and proved for $X = \mathbb{R}^n$ plays an important role in the geometry of convex sets. Other recent results related to the Helly theorem can be found in [2].

In [1], Chichilnisky provided some extensions of the Helly theorem and gave some applications of this theorem in economy. Then Kulpa [4] by using the Brouwer fixed point theorem strengthened this result.

In order to be more precise, let us introduce some notations. We shall use the notation $[p_0, \ldots, p_n] := conv\{p_0, \ldots, p_n\}$ for n-dimensional geometric simplex spanned by vertices p_i, where the points $p_0, \ldots, p_n$ are affinely independent. A k-dimensional simplex spanned by any $k+1$ of the vertices p_i of a simplex $S = [p_0, \ldots, p_n]$ is called a *k-face* of S. The union of all k-faces of the simplex S is called the *k-skeleton* of S. Also the $(n-1)$-skeleton of n-dimensional simplex S is said to be its geometric boundary ∂S.

A topological space X is k-connected, if each continuous map $f : \partial S \to X$ has a continuous extension over S; $F : S \to X$, $F|_{\partial S} = f$. If X is *k-connected* for each $k = 0, 1, \ldots,$ then X is said to be *∞-connected*.

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We will now discuss the terminology of the article. In Section 2, we introduce the generalized Helly family.

Definition 2.6. A family $\mathcal{C} = \{C_j\colon\ j \in \mathbb{Z}_+\}$ of subsets of a topological space X is said to be a generalized Helly family if the following holds:
For each finite subset $I \subset \mathbb{Z}_+$, the intersection $\bigcap_{i\in I} C_i$ is a nonempty ∞-connected set.
For each infinite proper subset $I \subset \mathbb{Z}_+$, the intersection $\bigcap_{i\in I} C_i$ is nonempty.

Then we extend the lemma on indexed covering for an infinite dimensional simplex to obtain the following result.

Theorem 2.12. *Suppose $\mathcal{C} = \{C_j\colon\ j \in \mathbb{Z}_+\}$ is a generalized Helly family of compact subsets of a topological space X. Then for each open (closed) covering $\mathcal{U} = \{U_j\colon\ j \in \mathbb{Z}_+\}$ of X such that $C_j \subset U_j$ for each $j \in \mathbb{Z}_+$, the intersection $\bigcap \mathcal{U}$ is a nonempty set.*

The previous result allows us to deduce the following theorem.

Theorem 2.13. *Let $\mathcal{C} = \{C_j\colon\ j \in \mathbb{Z}_+\}$ be a generalized Helly family of compact subsets of a topological space X. Then for each open (closed) covering $\mathcal{U} = \{U_j\colon\ j \in \mathbb{Z}_+\}$ of X such that $C_j \cap U_j = \emptyset$, for each $j \in \mathbb{Z}_+$, the intersection $\bigcap \mathcal{U}$ is a nonempty set.*

The next corollary is an immediate consequence of the above theorem.

Corollary 2.14. *If $\{A_j\colon\ j \in \mathbb{Z}_+\}$ is an open (closed) covering of infinite dimensional simplex Δ_0^∞ such that $A_j \cap S_j = \emptyset$, where $S_j = \{x \in \Delta_0^\infty\colon\ \forall i > j,\ x_i = 0\}$, for $j \in \mathbb{Z}_+$, then the intersection $\bigcap_j A_j$ is nonempty.*

In [3], Idczak and Majewski provided a generalization of the classical Poincaré–Miranda theorem [5] to the case of a denumerable set of continuous functions of denumerable number of variables.

In Section 3, we explore some consequences of generalized Poincaré–Miranda theorem. Suppose that $I^\infty = [0,1] \times [0,1] \times \cdots$ is the infinite dimensional cube of $\mathbb{R}^\infty$ and

$$I_i^- = \{x \in I^\infty\colon\ x_i = 0\}, \qquad I_i^+ = \{x \in I^\infty\colon\ x_i = 1\}.$$

Theorem 3.2. *If maps $g, h : I^\infty \to I^\infty$ are continuous and if $h(I_i^-) \subset I_i^-$ and $h(I_i^+) \subset I_i^+$ for each $i \in \mathbb{N}$, then there exists a point $c \in I^\infty$ such that $g(c) = h(c)$. Moreover, any continuous map $g : I^\infty \to I^\infty$ has a fixed point.*

Let Δ_0^∞ be the infinite dimensional simplex. Let us mention that Δ_0^∞ is homeomorphic to the Hilbert cube $\mathcal{H}$ [7]. So we conclude the following result.

Corollary 3.3. *If $g : \Delta_0^\infty \to \Delta_0^\infty$ is a continuous map then g has a fixed point.*

The following theorem is an extension of non-squeezing theorem [5] to infinite dimensional case.

Theorem 3.7. *Let $X = \varprojlim\{X_n, q_n\}$ be an inverse limit metric space and $h : I^\infty \to X$ be a continuous map onto X such that $h(I_i^-) \cap h(I_i^+) = \emptyset$ for each $i \in \mathbb{N}$. We take $h_n := h|_{I^n}$. Also let $h_n(I^n) \subset X_n$ and the following diagram commutes:*

$$\begin{array}{ccc} I^{n+1} & \xrightarrow{h_{n+1}} & X_{n+1} \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle q_n} \\ I^n & \xrightarrow[h_n]{} & X_n \end{array}$$

Then X is infinite dimensional.

2. Intersection properties of generalized Helly families

Let $\mathbb{R}^\infty$ be the product of countably many copies of $\mathbb{R}$. We equip $\mathbb{R}^\infty$ with the standard product topology, which is metrizable by the complete metric

$$\bar{d}(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)},$$

see [7]. The standard n-simplex in $\mathbb{R}^{n+1}$, denoted Δ_n, is the convex hull of the $n+1$ standard basis vectors of $\mathbb{R}^n$.