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Some observations on compact indestructible spaces

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ABSTRACT

Inspired by a recent work of Dias and Tall, we show that a compact indestructible space is sequentially compact. We also prove that a Lindelöf T_2 indestructible space has the finite derived set property and a compact T_2 indestructible space is pseudoradial. Finally, we observe that under CH a compact weakly Whyburn space of countable tightness is indestructible.

A compact space is indestructible if it remains compact in any countably closed forcing extension. This is a particular case of the notion of Lindelöf indestructibility, whose study was initiated by Tall in [14]. Since countable compactness is preserved in any countably closed forcing, a space is compact indestructible if and only if it is compact and Lindelöf indestructible. A nice connection of Lindelöf indestructibility with a certain infinite topological game was later discovered by Scheepers and Tall [12] (see Proposition 1).

$\mathrm{G}_1^{\omega_1}(\mathcal{O},\mathcal{O})$ denotes the game of length ω_1 played on a topological space X by two players I and II in the following way: at the α-th inning player I chooses an open cover $\mathcal{U}_\alpha$ of X and player II responds by taking an element $U_\alpha \in \mathcal{U}_\alpha$. Player II wins if and only if $\{U_\alpha\colon \alpha < \omega_1\}$ covers X.

Proposition 1. *([12, Theorem 1]) A Lindelöf space X is indestructibly Lindelöf if and only if player I does not have a winning strategy in $\mathrm{G}_1^{\omega_1}(\mathcal{O},\mathcal{O})$.*

Recently, Dias and Tall [7] started to investigate the topological structure of compact indestructible spaces. In particular, they proved that a compact T_2 indestructible space contains a non-trivial convergent sequence [7, Corollary 3.4].

The main aim of this short note is to strengthen the above result, by showing that indestructibility actually gives even more than sequential compactness (Theorem 5). However, indestructibility forces a compact space to be sequentially compact in the absolute general case, that is by assuming no separation axiom (Theorem 2). The same proof, with minor changes, will show that a Lindelöf T_2 indestructible space has the finite derived set property (Theorem 3).

As usual, $A \subseteq^* B$ means $|A \setminus B| < \aleph_0$ (mod finite inclusion).

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Theorem 2. *Every compact indestructible space is sequentially compact.*

Proof. Let X be a compact indestructible space and assume that X is not sequentially compact. Our task is to show that in this case player I would have a winning strategy in the game $G_1^{\omega_1}(\mathcal{O},\mathcal{O})$. Fix a sequence $\langle a_n\colon n<\omega\rangle$ with no convergent subsequence and let $A_0=\{a_n\colon n<\omega\}$. For each $x\in X$ there is an open set U_x^1 such that $x\in U_x^1$ and $|A_0\setminus U_x^1|=\aleph_0$. The first move of player I is the open cover $\mathcal{U}_1=\{U_x^1\colon x\in X\}$. If player II responds by choosing $V_1\in\mathcal{U}_1$, then let $A_1=A_0\setminus V_1$. For each $x\in X$ there is an open set U_x^2 such that $x\in U_x^2$ and $|A_1\setminus U_x^2|=\aleph_0$. The second move of player I is the open cover $\mathcal{U}_2=\{U_x^2\colon x\in X\}$. If player II responds by choosing $V_2\in\mathcal{U}_2$, then let $A_2=A_1\setminus V_2$. Again, for each $x\in X$ player I chooses an open set U_x^3 such that $x\in U_x^3$ and $|A_2\setminus U_x^3|=\aleph_0$, and so on. In general, at the α-th inning the moves of the two players have defined a mod finite decreasing family $\{A_\beta\colon \beta<\alpha\}$ of infinite subsets of A_0. Player I fixes an infinite set $B_\alpha\subseteq A_0$ such that $B_\alpha\subseteq^* A_\beta$ for each $\beta<\alpha$. Then, player I plays $\mathcal{U}_\alpha=\{U_x^\alpha\colon x\in X\}$, where U_x^α is an open set such that $x\in U_x^\alpha$ and $|B_\alpha\setminus U_x^\alpha|=\aleph_0$. If the response of player II is $V_\alpha\in\mathcal{U}_\alpha$, then let $A_\alpha=B_\alpha\setminus V_\alpha$. At the end of the game, the set resulting from the moves of player II is the collection $\mathcal{V}=\{V_\alpha\colon 1\leqslant\alpha<\omega_1\}$. For any finite set of ordinals $\alpha_0,\dots,\alpha_m<\omega_1$, taking some $\beta<\omega_1$ such that $\alpha_i<\beta$ for $i\leqslant m$, we see that the infinite set A_β has a finite intersection with each V_{α_i} and therefore the subcollection $\{V_{\alpha_i}\colon i\leqslant m\}$ cannot cover X. Since $\mathcal{V}$ does not have finite subcovers, the compactness of X implies that the whole $\mathcal{V}$ cannot cover X. Thus, player I wins the game, in contrast with Proposition 1. □

Recall that a topological space X has the *finite derived set* (briefly FDS) property provided that every infinite set of X contains an infinite subset with at most finitely many accumulation points (see for instance [5]). Since in a T_2 space a convergent sequence has only one accumulation point, we see that if a T_2 space has a countable infinite set A violating the finite derived set property, then for each infinite set $B\subseteq A$ and each point $x\in X$ there must be an open set U_x such that $x\in U_x$ and $|B\setminus U_x|=\aleph_0$. Notice, however, that for this much less than T_2 is needed. For instance, it suffices for the space to be SC, namely that every convergent sequence together with the limit point is a closed subset (see [5]).

With this observation in mind, we can modify the above proof to get the following.

Theorem 3. *A Lindelöf T_2 indestructible space has the finite derived set property.*

Proof. Let X be a Lindelöf T_2 indestructible space and assume that X does not have the FDS property. As in the proof of Theorem 2, our task is to show that in this case player I would have a winning strategy in the game $G_1^{\omega_1}(\mathcal{O},\mathcal{O})$. Fix a countable infinite set $A\subseteq X$ witnessing the failure of the FDS property. Taking into account the paragraph before the theorem, for each infinite set $B\subseteq A$ and each $x\in X$ there is an open set U_x such that $x\in U_x$ and $|B\setminus U_x|=\aleph_0$. Now, the strategy of player I is exactly the same as that in the proof of Theorem 2. At the end of the game, the set resulting from the moves of player II is again the collection $\mathcal{V}=\{U_{g\restriction\alpha+1}\colon \alpha<\omega_1\}$. We claim that $\mathcal{V}$ cannot cover X. Otherwise, by the Lindelöfness of X, there should exist a countable set of ordinals $S\subseteq\omega_1$ such that the subcollection $\{U_{g\restriction\alpha+1}\colon \alpha\in S\}$ would cover X. Taking some $\beta<\omega_1$ such that $\alpha<\beta$ for each $\alpha\in S$, we see that the infinite set $A_{g\restriction\beta}$ has a finite intersection with $U_{g\restriction\alpha+1}$ for each $\alpha\in S$. But, this implies that the infinite set $A_{g\restriction\beta}$ does not have accumulation points in X, in contrast with the supposed failure of the FDS property in A. Thus, $\mathcal{V}$ cannot cover X and again player I wins the game. □

The above theorem provides new information on the topological structure of a Lindelöf indestructible space.

Notice that, in view of Theorem 6a of [14], Theorem 3 strengthens Theorem 1.10 of [2].

We continue by showing that for T_2 spaces Theorem 2 can be improved.

Proposition 4. *([7, Corollary 3.3]) A compact T_2 space which is not first countable at any point is destructible.*

Recall that a topological space X is pseudoradial provided that for any non-closed set $A\subseteq X$ there exists a well-ordered net $S\subseteq A$ which converges to a point outside A. For more on these spaces see [6].

Clearly every compact T_1 pseudoradial space is sequentially compact, but the converse may consistently fail [8].

Theorem 5. *Any compact T_2 indestructible space is pseudoradial.*

Proof. Let X be a compact T_2 indestructible space and let A be a non-closed subset. We may assume, without any loss of generality, $X=\overline{A}$. Let λ be the smallest cardinal such that there exists a non-empty closed G_λ-set $H\subseteq X\setminus A$. As X is indestructible, so is the subspace H. Hence, by Proposition 4, H is first countable at some point p. Clearly, $\{p\}$ is a G_λ-set in X and so there are open sets $\{U_\alpha\colon \alpha<\lambda\}$ satisfying $\{p\}=\bigcap\{U_\alpha\colon \alpha<\lambda\}$. Moreover, we may assume that $\bigcap\{U_\beta\colon \beta<\alpha\}=\bigcap\{\overline{U_\beta}\colon \beta<\alpha\}$ holds for each limit ordinal α. The minimality of λ ensures that for each $\alpha<\lambda$ we may pick a point $x_\alpha\in A\cap\bigcap\{U_\beta\colon \beta<\alpha\}$. The compactness of X implies that the well-ordered net $\{x_\alpha\colon \alpha<\lambda\}$ converges to p and we are done. □

Notice that the indestructibility of a compact T_2 space is stronger than pseudoradiality: the example in Section 3 of [7] is a compact T_2 pseudoradial space which is destructible. This example is actually a radial space (= every point in the closure