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Waist of the sphere for maps to manifolds

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We generalize the sphere waist theorem of Gromov and the Borsuk–Ulam type measure partition lemma of Gromov–Memarian for maps to manifolds.

1. Introduction

In [4,11] M. Gromov and Y. Memarian prove the “sphere waist theorem” for a continuous map from a sphere S^n to the Euclidean space $\mathbb{R}^m$, showing that the preimage of some point has “large ε-neighborhoods”. Here we generalize it for maps from the sphere to any m-dimensional manifold.

Let the sphere S^n be the standard unit sphere in $\mathbb{R}^{n+1}$. Denote the standard probabilistic measure on S^n by μ, denote by $U_\varepsilon(X)$ the ε-neighborhood of $X \subseteq S^n$ with respect to the standard metric on S^n.

Theorem 1.1. *Suppose $h : S^n \to M$ is a continuous map from the n-sphere to a topological m-manifold with $m \leqslant n$. In case $m = n$ let the homology map $h_* : H_n(S^n; \mathbb{F}_2) \to H_n(M; \mathbb{F}_2)$ be trivial. Then there exists a point $z \in M$ such that for any $\varepsilon > 0$*

$$\mu U_\varepsilon\big(h^{-1}(z)\big) \geqslant \mu U_\varepsilon S^{n-m}.$$

Here S^{n-m} is the $(n-m)$-dimensional equatorial subsphere of S^n, i.e. $S^{n-m} = S^n \cap \mathbb{R}^{n-m+1}$.

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In [4, Section 6] the generalizations of the waist theorem for maps to a manifold are discussed, but it seems that no proof is given there. In this paper we rely on the more rigorous proof from [11], use the geometrical and analytical part of that proof, establish the corresponding Borsuk–Ulam type theorem for maps to manifolds, and finally deduce Theorem 1.1.

Manifolds in this paper are in the topological sense, they may be open or closed.

2. Corresponding generalization of the Borsuk–Ulam theorem

Following [11], we are going to prove the corresponding analogue of the Borsuk–Ulam theorem first. Following [8], we will prove a Borsuk–Ulam type theorem that is more general than it is needed to establish Theorem 1.1.

Let us give some definitions. Consider a compact topological space X with a probabilistic Borel measure μ. Let $C(X)$ denote the set of continuous functions on X.

Definition 2.1. A finite-dimensional linear subspace $L \subset C(X)$ is called *measure separating*, if for any $f \neq g \in L$ the measure of the set

$$e(f, g) = \{x \in X\colon\ f(x) = g(x)\}$$

is zero.

In particular, if X is a manifold and the measure μ is given by integrating a decent, say L_1, function, then any finite-dimensional space of analytic functions on X is measure separating, because the sets $e(f, g)$ in this case have dimension $< n$ and therefore measure zero. The reader is advised to keep this example in mind in order to avoid thinking too generally about measures and the measure separation property of functions.

Now for any collection of q elements of a measure-separating subspace we define a partition of X. Suppose $F = \{u_1, \dots, u_q\} \subset C(X)$ is a family of functions such that $\mu(e(u_i, u_j)) = 0$ for all $i \neq j$. Obviously, the sets (some of them may be empty)

$$V_i = \{x \in X\colon\ \forall j \neq i\ u_i(x) \geqslant u_j(x)\}$$

have a zero measure overlap, and we state:

Definition 2.2. Under the above assumptions we denote the family of subsets $P(F) = \{V_i\}_{i=1}^q$, this is a partition of X.

Recall that in case u_i are linear functions on $\mathbb{R}^n$ the partition $P(F)$ is known as a *generalized Voronoi partition* or a *power diagram*. Note that if we consider the standard sphere $S^n \subseteq \mathbb{R}^{n+1}$, and homogeneous linear functions $F \subset C(\mathbb{R}^{n+1}) \subset C(S^n)$ (this is the case we need for Theorem 1.1), then $P(F)$ is always a partition into convex subsets of S^n, or a partition consisting of one set equal to the whole S^n.

Now we have to discuss the notion of a "center map", that is selecting centers for the partition sets V_i. In [11] the center map was selected in a specific way to make some conclusions about the measure of the intersection of V_i and an ε-ball around the center of V_i, see some details in Remark 5.2 below and in [11]. Here we describe the minimal requirements on the center map needed for the corresponding generalization of the Borsuk–Ulam theorem to hold. Suppose we work with a finite-dimensional linear subspace of functions $L \subset C(X)$ such that $P(F) = \{V_1, \dots, V_q\}$ is defined for all q-element subsets $F \subset L$. Assume there is a way to assign *centers* $c(V_1), \dots, c(V_q) \in X$ to those of $V_1, \dots, V_q$ that have nonempty interior. What we actually need from such a *center map* is the following:

- The centers $c(V_1), \dots, c(V_q)$ depend continuously on $F \subset L$, whenever they are defined.
- If we permute the elements of F (and therefore $V_1, \dots, V_q$) then the centers are permuted accordingly.

Now we are ready to state the generalization of [11, Theorem 3].

Theorem 2.3. *Suppose L is a measure-separating subspace of $C(X)$ of dimension $n+1$, $\mu_1, \dots, \mu_{n-m}$ ($n > m$) are absolutely continuous (with respect to the original measure on X) probabilistic measures on X. Let $q = p^\alpha$ be a prime power, $c(\cdot)$ be a center map, as defined above, for q-tuples in L, and*

$$h : X \to M$$

be a continuous map to an m-dimensional topological manifold. Suppose also that the cohomology map $h^* : H^i(M; \mathbb{F}_p) \to H^i(X; \mathbb{F}_p)$ is a trivial map for $i > 0$.

Then there exists a q-element subset $F \subset L$ such that for every $i = 1, \dots, n-k$ the partition $P(F)$ partitions the measure μ_i into q equal parts, and we also have

$$h(c(V_1)) = h(c(V_2)) = \dots = h(c(V_q))$$

for $\{V_1, \dots, V_q\} = P(F)$.