Topology and its Applications 160 (2013) 1621–1632

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The geometric realization of a simplicial Hausdorff space is Hausdorff

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ARTICLE INFO

Article history:
Received 2 June 2012
Received in revised form 15 June 2013
Accepted 17 June 2013

MSC:
18G30
05E45
54D10
54D50

Keywords:
Simplicial space
Simplicial category
Geometric realization
Hausdorff spaces

ABSTRACT

We show that the thin geometric realization of a simplicial Hausdorff space is Hausdorff. This proves a long-standing conjecture of Graeme Segal stating that the thin geometric realization of a simplicial k-space is a k-space.

1. Introduction

1.1. The main problem

In one of his many landmark papers [4], Graeme Segal introduced the geometric realization functor for simplicial spaces, which he called the "thin" realization functor in the subsequent article [3]. He claimed that the thin geometric realization of a simplicial space which is compactly-generated Hausdorff degreewise must be compactly-generated Hausdorff. However, while it is essentially obvious that the geometric realization of a simplicial compactly-generated space must be compactly-generated, the Hausdorff property is a whole different matter since cocartesian squares are implicit in the definition of the thin geometric realization and they are known to behave badly with respect to separation axioms. At the time of [4], Segal's claim was thought to be dubious and no convincing proof of it ever appeared in the literature. This difficulty brought some to turn away from k-spaces and work with weak-Hausdorff compactly-generated spaces instead. It is much easier indeed to show that the geometric realization of a compactly-generated weak-Hausdorff space is itself compactly-generated

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[1] This work was completed while the author was working on his PhD thesis at the Institut Galilée in Université Paris Nord.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.06.007

weak-Hausdorff (this can be done with little effort using the tools from Appendix A of [1]). In the following pages, we will prove that Segal was right after all!

1.2. Definitions and notation

In this paper, we will use the French notation for the sets of integers: $\mathbb{N}$ will denote the set of natural numbers (i.e. non-negative integers), and $\mathbb{N}^*$ the one of positive integers. Recall the simplicial category Δ whose objects are the ordered sets $[n]=\{0,1,\dots,n\}$ for $n\in\mathbb{N}$ and whose morphisms are the non-decreasing maps, with the obvious compositions and identities. All the morphisms are composites of morphisms of two types, namely the face morphisms

$$\delta_i^k:\quad \begin{cases} [k]\rightarrow[k+1] \\ j<i\mapsto j \\ j\geqslant i\mapsto j+1 \end{cases}$$

for $k\in\mathbb{N}$ and $i\in[k+1]$, and the degeneracy morphisms

$$\sigma_i^k:\quad \begin{cases} [k]\rightarrow[k-1] \\ j\leqslant i\mapsto j \\ j>i\mapsto j-1 \end{cases}$$

for $k\in\mathbb{N}^*$ and $i\in[k-1]$. See [2] for a comprehensive account.

There is (covariant) functor $\Delta^*:\Delta\rightarrow\mathrm{Top}$ which sends $[n]$ to the n-simplex $\Delta^n:=\{(t_i)_{0\leqslant i\leqslant n}\in\mathbb{R}_+^{n+1}:\ \sum_{i=0}^n t_i=1\}$, and any morphism $\delta:[n]\rightarrow[m]$ to

$$\Delta^*(\delta):\quad \begin{cases} \Delta^n\rightarrow\Delta^m \\ (t_i)_{0\leqslant i\leqslant n}\mapsto\left(\sum_{i\in\delta^{-1}(j)}t_i\right)_{0\leqslant j\leqslant m}. \end{cases}$$

A **simplicial space** is a contravariant functor $\Delta\rightarrow\mathrm{Top}$. Given such a functor, we set $A_n:=A([n])$ for any $n\in\mathbb{N}$. For $k\in\mathbb{N}$, we will write $d_i^k:=A(\delta_i^k)$ for $i\in[k+1]$ (the face maps of A), and $s_i^k:=A(\sigma_i^k)$ for $i\in[k-1]$ (the degeneracy maps of A). When no confusion is possible, we will simply write δ_i instead of δ_i^k, σ_i instead of σ_i^k, d_i instead of d_i^k and s_i instead of s_i^k. If δ is a morphism in Δ, we will also write δ^* instead of $A(\delta)$.

For $n\in\mathbb{N}$, a point $x\in A_n$ is said to be **degenerate** when in the image of some s_i.

Definition 1.1. The **thin geometric realization** of a simplicial space A, denoted by $|A|$, is the quotient space of $\coprod_{n\in\mathbb{N}}A_n\times\Delta^n$ under the relations $(x,\Delta^*(\delta)[y])\sim(A(\delta)[x],y)$, for $(m,n)\in\mathbb{N}^2$, $x\in A_m$, $y\in\Delta_n$ and $\delta\in\mathrm{Hom}_\Delta([n],[m])$.

For every $n\in\mathbb{N}$, we thus have a natural map

$$\pi_n: A_n\times\Delta^n\rightarrow|A|.$$

Remark 1. Since no homotopy group will be considered here, no confusion should be excepted from our using the notation π_n to designate the above map.

Our simple aim here is to prove the following theorem:

Theorem 1.1. *Let A be a simplicial space and assume that A_n is Hausdorff for each $n\in\mathbb{N}$. Then, $|A|$ is Hausdorff.*

The proof, although very technical, has a very straightforward basic strategy: we will give a general construction of “flexible” open neighborhoods for the points of $|A|$ (see Section 2 for the construction and Section 3 for the proof of openness), and then show that those neighborhoods may be used to separate points (Section 4). In the rest of the paper, A denotes an arbitrary simplicial space (no separation assumption will be made until Section 4).