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Closed-constructible functions are piecewise closed

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Dedicated to the memory of topologist and friend D. Ranchin

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ABSTRACT

A subset of a topological space X is *constructible* if it belongs to the smallest algebra of subsets that contains all open subsets of X.
We prove that if a continuous function $f : X \to Y$ between separable metrizable spaces maps closed subsets of X onto constructible sets of Y, then X admits a countable closed cover $\mathcal{C}$ such that for each $C \in \mathcal{C}$ the restriction $f|C$ is closed.

The present paper continues the series of publications about decomposibility of Borel functions[1] f between separable metrizable spaces into a finite or countable number of open, closed and continuous functions.

In the first publication [7] we have raised the question about preservation of complete metrizability by some simplest Borel functions. The author has limited his first work to only one question about the preservation of completeness, since it was not clear whether such a decomposition was possible. It is known, however, that if a decomposition is possible, it leads to preservation of even other Borel classes [9]. Recently, this question was fully and affirmatively resolved by Gao and Kleinfield [1], Holický and Pol [3].

In the following publications [8,10] we have obtained such a decomposition into open, closed and continuous functions in the case of simplest Borel functions.

In the present paper we give an affirmative solution for the first case when images of closed sets[2] are unrestricted combinations of closed and open sets.

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[1] f is Borel in a broad meaning of the term: f and f^{-1} are multivalued functions that map open or closed sets onto Borel ones.
[2] The case where they are open will be investigated in the next publication of the series: "Open-constructible functions".

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A subset of a topological space X is *constructible* if it belongs to the smallest algebra of subsets that contains all open subsets of X. Equivalently, a subset $A \subset X$ is constructible if it can be written as a finite union of locally closed sets. A set is *locally closed* if it is an intersection of a closed and an open set.

The main result of this paper is as follows:

Theorem 1. *If a continuous function $f : X \to Y$ between separable metrizable spaces maps closed subsets of X onto constructible sets of Y, then X admits a countable closed cover $\mathcal{C}$ such that for each $C \in \mathcal{C}$ the restriction $f|C$ is closed.*

Proof. This result follows from the fact that the set Z in Lemma 1 is empty. Indeed, if we suppose otherwise, then the restriction $f|Z$ is nowhere closed on Z and maps closed sets onto constructible sets. This contradicts Lemma 3. □

In order to prove the lemmas below, we need to introduce two key definitions:

- A function $f : X \to Y$ is *nowhere closed* on X if for every nonempty open set $A \subset X$ the restriction $f|cl_X A$ is not closed.
- A function $f : X \to Y$, for which the space X admits a countable closed cover $\mathcal{C}$ such that for each $C \in \mathcal{C}$ the restriction $f|C$ is closed, is often called *piecewise closed*.

Lemma 1. *For every map $f : X \to Y$ between separable metrizable spaces there is a closed subset $Z \subset X$ such that the restriction $f|Z$ is nowhere closed on Z and the restriction $f|(X \setminus Z)$ is piecewise closed.*

Proof of Lemma 1. Let us begin by proving the first part of the assertion from Lemma 1 stating that for some Z the restriction $f|Z$ is nowhere closed on Z. Indeed, if for some nonempty open set $V \subset X$ the restriction $f|cl_X V$ is closed, then we could construct the closed set

$$X_1 = X \setminus V$$

and the corresponding restriction

$$f|X_1 : X_1 \to f(X_1).$$

Repeating this process, we could also construct a chain of closed sets ($X_\gamma = \bigcap_{\beta<\gamma} X_\beta$ for a limit γ)

$$X \supset X_1 \supset \cdots \supset X_\gamma \supset \cdots$$

which, as we know, stabilizes at some $\gamma_0 < \omega_1$. Therefore, there exists a subspace Z for which holds true

$$Z = X_{\gamma_0} = X_{\gamma_0+1} = \cdots$$

and the restriction $f|Z$ is nowhere closed on Z.

In order to prove the second part of Lemma 1 stating that $f|(X\setminus Z)$ is piecewise closed, we shall note that according to our construction each difference $X_\gamma \setminus X_{\gamma+1}$ is open in X_γ, hence F_σ in X_γ and each restriction $f|cl_{X_\gamma}(X_\gamma \setminus X_{\gamma+1})$ is closed. Since each X_γ is closed in X, it implies that each $f|(X_\gamma \setminus X_{\gamma+1})$ and hence $f|(X \setminus Z)$ is piecewise closed. □

Lemma 2. *Let $f : X \to Y$ be a continuous nowhere closed on X function between separable metrizable spaces. Then for each open in X subset V, point $x \in V$, open neighbourhood $O(y)$ of the point $y \in f(x)$ and*