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Approximation and interpolation by large entire cross-sections of second category sets in $\mathbb{R}^{n+1}$ ☆

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In [M.R. Burke, Large entire cross-sections of second category sets in $\mathbb{R}^{n+1}$, Topology Appl. 154 (2007) 215–240], a model was constructed in which for any everywhere second category set $A \subseteq \mathbb{R}^{n+1}$ there is an entire function $f : \mathbb{R}^n \to \mathbb{R}$ which cuts a large section through A in the sense that $\{x \in \mathbb{R}^n\colon (x, f(x)) \in A\}$ is everywhere second category in $\mathbb{R}^n$. Moreover, the function f can be taken so that its derivatives uniformly approximate those of a given C^N function g in the sense of a theorem of Hoischen. In the theory of the approximation of C^N functions by entire functions, it is often possible to insist that the entire function interpolates the restriction of the C^N function to a closed discrete set. In the present paper, we show how to incorporate a closed discrete interpolation set into the above mentioned theorem. When the set being sectioned is sufficiently definable, an absoluteness argument yields a strengthening of the Hoischen theorem in ZFC. We get in particular the following: Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is a C^N function, $\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is a positive continuous function, $T \subseteq \mathbb{R}^n$ is a closed discrete set, and $G \subseteq \mathbb{R}^{n+1}$ is a dense G_δ set. Let $A \subseteq \mathbb{R}^n$ be a countable dense set disjoint from T and for each $x \in A$, let $B_x \subseteq \mathbb{R}$ be a countable dense set. Then there is a function $f : \mathbb{R}^n \to \mathbb{R}$ which is the restriction of an entire function $\mathbb{C}^n \to \mathbb{C}$ such that the following properties hold. (a) For all multi-indices α of order at most N and all $x \in \mathbb{R}^n$, $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$, and moreover $(D^\alpha f)(x) = (D^\alpha g)(x)$ when $x \in T$. (b) For each $x \in A$, $f(x) \in B_x$. (c) $\{x \in \mathbb{R}^n\colon (x, f(x)) \in G\}$ is a dense G_δ set in $\mathbb{R}^n$.

1. Introduction

In this paper, we extend the work begun in [2].

1.1. Approximating smooth functions with entire functions

A theorem of Carleman [6], extending the well-known theorem of Weierstrass on approximation by polynomials of continuous functions on compact intervals, states that for every continuous function $f : \mathbb{R} \to \mathbb{R}$

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and every positive continuous function $\varepsilon : \mathbb{R} \to \mathbb{R}$, there is a function $g : \mathbb{R} \to \mathbb{R}$ which is the restriction to $\mathbb{R}$ of an entire function and satisfies $|g(x) - f(x)| < \varepsilon(x)$ for all $x \in \mathbb{R}$. L. Hoischen proved the following generalization which allows approximation of both f and its derivatives. In this paper, t denotes a positive integer.

Theorem 1.1. *([11], see also [9].) Let $g : \mathbb{R}^t \to \mathbb{C}$. Let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function.*

(1) *Let $k \geqslant 0$ be an integer. If g is a C^k function then there is an entire function $f : \mathbb{C}^t \to \mathbb{C}$ such that $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ when $x \in \mathbb{R}^t$, $|\alpha| \leqslant k$.*
(2) *If g is a C^∞ function then for each sequence $0 \leqslant c_0 \leqslant c_1 \leqslant \cdots$ of real numbers such that $\lim_{k\to\infty} c_k = \infty$, there is an entire function $f : \mathbb{C}^t \to \mathbb{C}$ such that, for all $k = 0, 1, 2, \ldots,$ $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ when $x \in \mathbb{R}^t$, $|x| \geqslant c_k$, and $|\alpha| \leqslant k$.*

In both of these statements, if g is real-valued then we may require that f takes real values on $\mathbb{R}^t$.

For the case $t = 1$, this theorem is improved in [12] to give simultaneously approximation of the derivatives $D^n g$ of a smooth function g as well as interpolation of the restriction of the derivatives to a closed discrete set. Hoischen's method can be adapted to functions of several variables. The adaptation requires some effort, but it has been carefully written out in [5].

Theorem 1.2. *([12], see also [5].) Let $g : \mathbb{R}^t \to \mathbb{C}$. Let $T \subseteq \mathbb{R}^t$ be a closed discrete set and let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function.*

(1) *Let $k \geqslant 0$ be an integer. If g is a C^k function then there exists an entire function $f : \mathbb{C}^t \to \mathbb{C}$ such that*
   (a) *$|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ for $x \in \mathbb{R}^t$, $|\alpha| \leqslant k$, and*
   (b) *$(D^\alpha f)(x) = (D^\alpha g)(x)$ whenever $x \in T$, $|\alpha| \leqslant k$.*
(2) *If g is a C^∞ function then for each sequence $0 \leqslant c_0 \leqslant c_1 \leqslant \cdots$ of real numbers such that $\lim_{k\to\infty} c_k = \infty$ there exists an entire function $f : \mathbb{C}^t \to \mathbb{C}$ such that for all $k = 0, 1, 2, \ldots$*
   (a) *$|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ for $x \in \mathbb{R}^t$, $|x| \geqslant c_k$, $|\alpha| \leqslant k$, and*
   (b) *$(D^\alpha f)(x) = (D^\alpha g)(x)$ whenever $x \in T$, $|x| \geqslant c_k$, $|\alpha| \leqslant k$.*

In both of these statements, if g is real-valued then we may require that f takes real values on $\mathbb{R}^t$.

As a device for simplifying such statements, which we need to make repeatedly throughout the paper, we introduce the following notation.

Definition 1.3. The set $\mathcal{S}$ and the relations $f <^k_T \varepsilon$ and $f <_{\bar{c},T} \varepsilon$ are defined as follows. The function f here is real- or complex-valued, defined on $\mathbb{R}^t$ or $\mathbb{C}^t$ and sufficiently differentiable for the derivatives in the definition to exist. (In this paper, we always have $f[\mathbb{R}^t] \subseteq \mathbb{R}$ except in the statements of Hoischen's theorems above.) $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ is a positive function.

1. $\mathcal{S}$ denotes the set of all sequences $\bar{c} = (c_i\colon\ i < \omega)$ of real numbers such that $0 = c_0 \leqslant c_1 \leqslant \cdots$, $\lim_{i\to\infty} c_i = \infty$.
2. For $k < \omega$, $f <^k_T \varepsilon$ means that for all $x \in \mathbb{R}^t$ and all multi-indices α such that $|\alpha| \leqslant k$, we have
   (a) $|(D^\alpha f)(x)| < \varepsilon(x)$;
   (b) $(D^\alpha f)(x) = 0$ when $x \in T$.
   $f <^k \varepsilon$ means $f <^k_\emptyset \varepsilon$.
3. $f <_{\bar{c},T} \varepsilon$ means that for all $x \in \mathbb{R}^t$, all multi-indices α, and all nonnegative integers k, if $|x| \geqslant c_k$ and $|\alpha| \leqslant k$ then