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Indecomposability in inverse limits with set-valued functions

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ABSTRACT

In this paper, we develop a sufficient condition for the inverse limit of upper semi-continuous functions to be an indecomposable continuum. This condition generalizes and extends those of Ingram and Varagona. Additionally, we demonstrate a method of constructing upper semi-continuous functions whose inverse limit has the full projection property.

1. Introduction

A topological space X is a *continuum* if it is a non-empty compact, connected, metric space. Unless otherwise specified, all spaces in this paper are assumed to be continua. For a continuum X, we denote the collection of non-empty compact subsets of X by 2^X. A continuum Y which is a subset of X is a subcontinuum of X. For brevity, we will use $Y \trianglelefteq X$ to indicate that Y is a subcontinuum of X.

If X and Y are continua, a function $f : X \to 2^Y$ is *upper semi-continuous at* x provided that for all open sets V in Y which contain $f(x)$, there exists an open set U in X with $x \in U$ such that if $t \in U$, then $f(t) \subseteq V$. If $f : X \to 2^Y$ is upper semi-continuous at each $x \in X$, we say that f is *upper semi-continuous* (*usc*). A usc function $f : X \to 2^Y$ is called *surjective* provided that for each $y \in Y$, there exists $x \in X$ with $y \in f(x)$.

The *graph* of a function $f : X \to 2^Y$ is the subset $G(f)$ of $X \times Y$ for which $(x, y) \in G(f)$ if and only if $y \in f(x)$. Ingram and Mahavier showed that a function $f : X \to 2^Y$ is usc if and only if $G(f)$ is closed in $X \times Y$ [4]. This condition is easier to verify than the definition, and will be used frequently throughout the

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paper. As a consequence, it is easy to check that if $f : X \to 2^Y$ is usc and surjective, then the *inverse of* f, ($f^{-1} : Y \to 2^X$) defined by $x \in f^{-1}(y)$ if and only if $y \in f(x)$ is well-defined and usc.

As a generalization of the well-studied and well-understood theory of inverse limits on continua, Ingram and Mahavier introduced the notion of inverse limits with set-valued functions [4,5].

Definition 1. Let $X_1, X_2, \dots$ be a sequence of continua and for each $i \in \mathbb{N}$, let $f_i : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous function. The *inverse limit of the pair* $\{X_i, f_i\}$ is the set

$$\varprojlim\{X_i, f_i\} = \big\{(x_i)_{i=1}^{\infty}\colon\ x_i \in f_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\big\}$$

with the topology inherited as a subset of the product space $\prod_{i=1}^{\infty} X_i$.

The spaces X_i are called the *factor spaces* of the inverse limit, and the usc functions f_i the *bonding functions*. An *inverse sequence* $\{X_i, f_i\}$ denotes sequences X_i and f_i for which $f_i : X_{i+1} \to 2^{X_i}$ is a usc function. We will use $\pi_j : \varprojlim\{X_i, f_i\} \to X_j$ to denote the restriction to $\varprojlim\{X_i, f_i\}$ of the usual projection map on $\prod_{i=1}^{\infty} X_i$. For a subset L of the natural numbers, we will use π_L to denote projection from $\varprojlim\{X_i, f_i\}$ to $\prod_{i\in L} X_i$. We will often use the notation $[k, n]$ to denote the subset of the natural numbers consisting of k, n and all natural numbers between them.

As this is a generalization of the usual notion of inverse limits with continuous functions as bonding maps, it is natural to investigate the extent to which results carry over into this new context. Ingram and Mahavier pioneered this inquiry [4,5], and many others have continued the investigation. We state some of these results below.

Theorem 2. *([4]) Let $\{X_i, f_i\}$ be an inverse sequence. Then $\varprojlim\{X_i, f_i\}$ is non-empty and compact.*

It is well-known that if the bonding maps in an inverse sequence are continuous functions, the resulting inverse limit will be connected. This is, however, not the case in the context of usc functions. In particular, examples of non-connected inverse limits may be found in [4,5] among others. The following known results indicate some sufficient conditions for an inverse limit of usc functions to be connected.

Theorem 3. *([4]) Let $\{X_i, f_i\}$ be an inverse sequence and suppose that for each $i \in \mathbb{N}$ and each $x \in X_{i+1}$, $f_i(x)$ is connected. Then $\varprojlim\{X_i, f_i\}$ is connected.*

Theorem 4. *([4]) Let $\{X_i, f_i\}$ be an inverse sequence and suppose that for each $i \in \mathbb{N}$ and each $x \in X_i$, $f_i^{-1}(x)$ is non-empty and connected. Then $\varprojlim\{X_i, f_i\}$ is connected.*

The following theorem is as stated in [3]. A more general version can be found in [8].

Theorem 5. *([8]) Suppose $\mathcal{F}$ is a collection of usc functions from $[0,1]$ to $2^{[0,1]}$ such that for every $g \in \mathcal{F}$ and $x \in [0,1]$, $g(x)$ is connected, and that f is the function whose graph is the union of all the graphs of the functions in $\mathcal{F}$. If f is surjective and $G(f)$ is a continuum, then $\varprojlim f$ is a continuum.*

For functions defined as in this theorem, we will write, $f = \bigcup_{g\in\mathcal{F}} g$. The following theorem will also be useful.

Theorem 6. *([8]) Suppose X is a compact Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semi-continuous set-valued function. Then $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$ is connected.*

In their development of connectedness theorems, Ingram and Mahavier use a generalized notion of the graph of a function. We will use the following notation for this idea in this paper.