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On strong size levels

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ABSTRACT

H. Hosokawa introduced the concept of a strong size property as a generalization of a Whitney property. In this paper we determine whether some properties are either m-strong size properties, m-strong size-reversible properties or m-sequential strong size-reversible properties.

1. Introduction

The hyperspace of compact subsets of a metric space X is defined by $2^X = \{A \subset X\colon\ A$ is compact and nonempty$\}$ and is equipped with the Hausdorff metric H [7, 2.1, p. 11]. For $m \in \mathbb{N}$ the m-fold hyperspace of a continuum X is defined by $C_m(X) = \{A \in 2^X\colon\ A$ has at most m components$\}$ and the m-fold symmetric product of X is given by $F_m(X) = \{A \in 2^X\colon\ A$ has at most m elements$\}$; both are regarded as subspaces of 2^X.

H. Hosokawa introduced in [5] the following generalization of a Whitney map for a fixed $m \in \mathbb{N}$: a *strong size map* for $C_m(X)$ is a map $\sigma : C_m(X) \to \mathbb{R}$ such that $\sigma(A) = 0$ whenever $A \in F_m(X)$ and $\sigma(A) < \sigma(B)$ whenever $A \subsetneq B$ and $B \notin F_m(X)$. Furthermore, a *strong size level* for $C_m(X)$ is a set of the form $\sigma^{-1}(t)$ for some $t \in [0, \sigma(X)]$. Hosokawa showed that strong size levels are continua [5, Theorem 2.10, p. 958] and defined a topological property P to be a *strong size property* provided that if a continuum X has property P,

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so does $\sigma^{-1}(t)$ for each strong size map σ for $C_m(X)$ for each $t \in [0, \sigma(X))$ and for each $m \in \mathbb{N}$ [5, p. 962]. Hence, strong size properties are a natural generalization of Whitney properties.

Many authors have studied Whitney properties, Whitney-reversible properties and sequential strong Whitney-reversible properties (for a comprehensive discussion see Chapter VIII of [7]). Thus, it is only natural to study strong size properties, for instance in [5, Theorems 3.1, 3.3 and 3.4] it is proved that local connectedness, arcwise connectedness and aposyndesis are strong size properties.

In this paper we say that a topological property P is an *m-strong size property* provided that if a continuum X has property P, so does $\sigma^{-1}(t)$ for each strong size map σ for $C_m(X)$ and for each $t \in [0, \sigma(X))$. Thus, strong size properties are m-strong size properties for all $m \in \mathbb{N}$. Moreover, P is an *m-strong size-reversible property* provided that whenever X is a continuum such that $\sigma^{-1}(t)$ has property P for all strong size maps σ for $C_m(X)$ and all $t \in [0, \sigma(X))$, then X has property P. Finally, we say that P is an *m-sequential strong size-reversible property* provided that whenever X is a continuum such that there is a strong size map σ for $C_m(X)$ and a sequence $(t_n)_{n=1}^{\infty}$ in $(0, \sigma(X))$ such that $\lim t_n = 0$ and $\sigma^{-1}(t_n)$ has property P for each $n \in \mathbb{N}$, then X has property P.

Recall that containing R^3-sets (defined in Preliminaries) is one of the most useful conditions for non-contractibility of spaces and hyperspaces. In [7, Question 58.3, p. 287] it is asked whether the property of containing no R^3-set is a Whitney property; we show that the property of containing no R^3-set is not an m-strong size property for any $m \in \mathbb{N}$. On the other hand, it is not known whether the property of containing an R^3-continuum is Whitney-reversible [7, Question 58.4, p. 286]; here we prove that it is not an m-sequential strong size-reversible property for any $m \in \mathbb{N}$ (Corollary 3.5). We show a similar result for R^3-sets (Corollary 4.5). We also show that none of the following properties is an m-strong size property for any $m \in \mathbb{N}$: (a) not containing R^3-sets, (b) being contractible, (c) having contractible hyperspace 2^X and/or $C_k(X)$ and (d) being arc-smooth (Corollary 4.6).

Finally, it is known that irreducibility is not a Whitney property [3, Example 3.2] and in [7, Question 49.9, p. 276] it is asked if there exists an irreducible continuum X such that every positive Whitney level is not irreducible (equivalently, if the property of not being irreducible is a Whitney-reversible property). Such an example was actually constructed earlier in [8, Example 5.4, p. 178], however that continuum was indecomposable. Thus, the question remained as if there exists an irreducible, hereditarily decomposable continuum such that every positive Whitney level is not irreducible. In Section 5 we give a positive answer to this question and, more generally, we give an example of an irreducible, hereditarily decomposable continuum such that every positive strong size level is not irreducible.

This paper is organized in five sections. After Introduction and Preliminaries, in Section 3 we introduce our results connected with R^3-continua. In Section 4 we present our results on R^3-sets and contractibility. Finally, in Section 5 we present our example about irreducibility and we conclude with some open questions.

2. Preliminaries

The symbol $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{R}^m$ denotes the m-dimensional Euclidean space. All considered spaces are assumed to be metric, and all maps are continuous functions.

The symbols $\operatorname{int}_Y(A)$, $\overline{A}^Y$ and $\operatorname{bd}_Y(A)$ stand for the topological interior, closure and boundary of the set A relative to the subspace Y of a space X, respectively. In case $X = Y$ we will simply omit the symbol Y. Also, $\operatorname{diam}(A)$ will denote the diameter of a (metric) space A, $|A|$ will denote its cardinality and $\mathcal{P}(A)$ will denote its power set. Moreover, for a positive number ε and a point p of a metric space (X, d) we define $B(\varepsilon, p) = \{x \in X\colon\ d(p, x) < \varepsilon\}$.

Throughout the paper we will use the following notation (which is taken from [6]): if $p_1, \dots, p_n$ are points in $\mathbb{R}^m$, we denote by $p_1 \cdots p_n$ the broken line with vertices $p_1, \dots, p_n$. In particular p_1p_2 denotes the segment that joins p_1 and p_2. Further, if $t \in \mathbb{R}$, $p \in \mathbb{R}^m$ and $S \subset \mathbb{R}^m$ then tS denotes the set $\{tq \in \mathbb{R}^m\colon\ q \in S\}$, pS denotes the set $\bigcup\{pq \subset \mathbb{R}^m\colon\ q \in S\}$ and $p + S$ denotes the set $\{p + q \in \mathbb{R}^m\colon\ q \in S\}$.