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The revised and uniform fundamental groups and universal covers of geodesic spaces

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ABSTRACT

Sormani and Wei proved in 2004 that a compact geodesic space has a categorical universal cover if and only if its covering/critical spectrum is finite. We add to this several equivalent conditions pertaining to the geometry and topology of the revised and uniform fundamental groups. We show that a compact geodesic space X has a universal cover if and only if the following hold: 1) its revised and uniform fundamental groups are finitely presented, or, more generally, countable; 2) its revised fundamental group is discrete as a quotient of the quasitopological fundamental group $\pi_1^{qtop}(X)$. In the process, we classify the topological singularities in X, and we show that the above conditions imply closed liftings of all sufficiently small path loops to all covers of X, generalizing the traditional semilocally simply connected property. A geodesic space X with this new property is called semilocally r-simply connected, and X has a universal cover if and only if it satisfies this condition. We then introduce the covering topology on $\pi_1(X)$, which can be considered a geometrization of both Brazas–Fabel's shape topology and the topology induced by the more general Spanier groups. We establish several connections between properties of the covering topology, the existence of simply connected and universal covers, and geometries on the fundamental group.

1. Introduction and main results

In [32], Sormani and Wei formally defined the covering spectrum of a compact geodesic space, a geometric invariant that detects one-dimensional holes of positive intrinsic diameter. They showed (Theorem 3.4, [32]) that a compact geodesic space X has a universal cover if and only if its covering spectrum, $CovSpec(X)$, is finite. When this holds, they defined the revised fundamental group of X to be the deck group of the universal cover, and they showed that it is finitely generated (Proposition 6.4, [32]).

In this paper, we extend the above results through an investigation of the geometry and topology of a slightly generalized revised fundamental group and another associated group called the uniform fundamental group. To do so, we apply the generalized covering methods developed by Berestovskii and Plaut for uniform spaces [2] to the narrower but important class of compact geodesic spaces.

In [2], Berestovskii and Plaut defined the uniform universal covering and its deck group, the uniform fundamental group. These are generalizations of the classical universal cover and fundamental group for uniform spaces – hence, metric spaces – that are not necessarily semilocally simply connected or even locally path connected. Spaces for which the uniform universal cover exists are called coverable, and these include all geodesic spaces and, thus, Gromov–Hausdorff limits of Riemannian manifolds. The foundation for [2] is discrete homotopy theory, an analog of classical path homotopy theory that uses discrete chains and chain homotopies instead of their continuous path counterparts. In [36], and with Plaut et al. in [14], the

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author used discrete homotopy theory to generalize the covering spectrum. When the methods of [2] are applied to a metric space X, one obtains the $\mathbb{R}_+$-parameterized collection of ε-covers of X, $\{X_\varepsilon\}_{\varepsilon>0}$. These covers, in turn, determine the critical spectrum of X, the set of values, $Cr(X)$, at which the equivalence class of X_ε changes as ε decreases to 0. The uniform universal cover and uniform fundamental group are inverse limits of the ε-covers and their deck groups, respectively.

With the exception of the inverse limit formulations, this construction and spectral definition parallel those of Sormani and Wei in [31] and [32]. The primary difference between the covering and critical spectra is the applicability. The Sormani–Wei construction relies on a classical method of Spanier [34] that requires local path connectivity of the underlying metric space X, which – if X is compact and connected – is equivalent to X being geodesic. The Berestovskii–Plaut construction, however, can be carried out much more generally, allowing investigation of the critical spectra of more exotic and pathological metric spaces. Like the covering spectrum, the critical spectrum detects fundamental group generators, but it also detects other metric structures in the general case that do not show up in geodesic spaces (cf. [14]). Nevertheless, Plaut and the author showed in [29] that when the underlying metric space is compact geodesic, the two spectra differ only by a constant multiple, namely $3\,Cr(X) = 2\,CovSpec(X)$. Thus, the covering spectrum, appropriately rescaled, is a special case of the critical spectrum in the compact geodesic setting. In particular, this fact and Sormani–Wei's theorem, together, show that a compact geodesic space has a universal cover if and only if its critical spectrum is finite.

We can now outline our major results. Since we will be exploiting the Berestovskii–Plaut uniform methods, our results and proofs will be presented in the language of discrete homotopy theory and the critical spectrum, the relevant technical background for which is given in Section 2. In this paper, a cover of X will always imply a traditional, connected cover $f : Y \to X$ with the property that each $x \in X$ is contained in an evenly covered neighborhood with respect to f. A *universal cover* of X will mean a traditional, categorical universal cover (not necessarily simply connected), or a cover $f : Y \to X$ so that, for any other cover $g : Z \to X$, there is a cover $h : Y \to Z$ such that $g \circ h = f$. Except for the uniform universal cover, we will not need or use any of the recent, non-traditional generalizations of universal covers that relax the evenly covered property (cf. [6,7,21,26]). When we use the uniform universal cover, it will always be explicitly referenced as such, so no confusion should result.

In Section 3 we slightly generalize the revised fundamental group defined by Sormani and Wei in [32]. The normal covering groups of the ε-covers, $\{K_\varepsilon\}$, intersect to form the *closed lifting group*, the normal subgroup $\pi_{cl}(X) \trianglelefteq \pi_1(X)$ representing all loops at the base point $*$ that lift closed (i.e. lift as loops) to every X_ε. The revised fundamental group is $\bar{\pi}_1(X) := \pi_1(X)/\pi_{cl}(X)$. It isomorphically injects into $\Delta(X)$, the uniform fundamental group of X, which, in turn, is isomorphic to $\check{\pi}_1(X)$, the first shape group of X. When X has a universal cover, $\bar{\pi}_1(X)$ agrees with Sormani–Wei's definition, though they only define this group in that particular case. Our approach shows that $\pi_{cl}(X)$ and $\bar{\pi}_1(X)$ are well-defined for any geodesic X. A specific property of $\pi_{cl}(X)$ determines when X has a universal cover (Lemma 4.1).

Two obvious cases of interest are when $\pi_{cl}(X)$ is trivial or all of $\pi_1(X)$. See Section 4 for examples. In the former case, $\bar{\pi}_1(X)$ is just $\pi_1(X)$, which then injects into $\Delta(X)$. Lemma 3.12 shows that $\pi_{cl}(X)$ is always trivial for some common classes of spaces, including one-dimensional spaces (see also Proposition 5.12). When $\pi_{cl}(X) = \pi_1(X)$, X is its own universal cover and $Cr(X) = \emptyset$. These conditions are actually equivalent, which we show in Corollary 4.9.

We also extend to $\bar{\pi}_1(X)$ some classical notions related to the fundamental group. We define X to be *semilocally r-simply connected* if each $x \in X$ has a neighborhood U such that every path loop in U based at x lifts closed to X_ε for all $\varepsilon > 0$ (Definition 3.15). This generalizes classical semilocally simple connectivity. Additionally, the classical space-fundamental group functor, $f \mapsto f_*$, has an analog for revised fundamental groups, and a homotopy equivalence $f : X \to Y$ induces an isomorphism $f_\sharp : \bar{\pi}_1(X) \to \bar{\pi}_1(Y)$ (Lemma 3.17 and Corollary 3.18).

Proposition 4.3 is the main technical result of the paper, showing that X is semilocally r-simply connected if $\bar{\pi}_1(X)$ is countable. We then use lifting properties of loops to classify the two basic types of topological singularities that obstruct semilocally simply connectedness (Definition 4.5). Roughly speaking, *sequentially singular points* capture the type of singularity one finds in the Hawaiian earring, while *degenerate points* generalize the failure of X to be homotopically Hausdorff at a point. Our first and primary theorem is

Theorem 1.1. *If X is a compact geodesic space, then the following are equivalent.*

1) *X has a universal cover.*
2) *$Cr(X) = \frac{2}{3}\,CovSpec(X)$ is finite.*
3) *The revised fundamental group, $\bar{\pi}_1(X)$, is any one of the following*: i) *countable*; ii) *finitely generated*; iii) *finitely presented.*
4) *The uniform fundamental group, $\Delta(X)$, is any one of the following*: i) *countable*; ii) *finitely generated*; iii) *finitely presented.*
5) *X has no sequentially singular points.*
6) *X is semilocally r-simply connected.*

If these hold for X, then its universal cover $\hat{X}$ is r-simply connected (*i.e. $\bar{\pi}_1(\hat{X})$ is trivial*)*, its deck and covering groups, respectively, are $\bar{\pi}_1(X)$ and $\pi_{cl}(X)$, and $\bar{\pi}_1(X)$ is isomorphic to $\Delta(X)$.*

We have already noted that $1 \Leftrightarrow 2$ is known. The proof of Theorem 1.1 will mostly show that the other statements are equivalent to 2, but we include 1 for both emphasis and reference. Moreover, the implication $2 \Rightarrow$ 3iii follows directly from