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A homotopically Hausdorff space which does not admit a generalized universal covering space

Žiga Virk[a], Andreas Zastrow[b,*]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
[b] Institute of Mathematics, Gdansk University, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

It is well known that every space which admits a generalized universal covering space is homotopically Hausdorff. The purpose of this paper is to construct a homotopically Hausdorff space, which does not admit this construction of a generalized universal covering space.

1. Introduction

Classical covering theory has been proven to be an immense useful tool on the edge of algebra and topology, mainly since it provides a representation of the fundamental group of a space by providing its action as group of deck-transformations on the simply connected (universal) covering space, and it provides the representation of the given space as quotient of this covering space via the action of this group. However, classical covering theory only works for semilocally simply connected spaces (cf. [20, p. 174], but also [16, Def. 2.1–2.2 & Thm. 2.8]).

Therefore in [17] a proposition was made to make the advantages of covering space theory available also outside the class of semilocally simply connected spaces. However, this theory does still not work for all spaces, and, roughly speaking, properties of homotopic Hausdorffness become essential to make this theory of "generalized covering spaces" work. The property "homotopically Hausdorff" (cf. [10, Def. 5.2] and Definition 1 below) was observed to be a necessary condition for the existence of generalized universal covering spaces, while "homotopically path-Hausdorff" was found as a sufficient condition (cf. below in the introduction). All spaces that were known not to admit a generalized universal covering space, were not homotopically Hausdorff, and thus it was (actually already since the mid-nineties, cf. the same place) open whether homotopically Hausdorff was not already sufficient to guarantee the existence of generalized universal covering spaces. In case of a countable fundamental group it was, as shown in [17, Thm. 4.4]. Since [16, §6(14)] it was also clear that the two versions of homotopic Hausdorffness were not equivalent. The example presented in this paper now settles this open question in the negative way: in general the property "homotopical Hausdorff" does not suffice to guarantee the existence of a generalized universal covering space (Thm. 16).

* Corresponding author.
E-mail addresses: zigavirk@gmail.com (Ž. Virk), zastrow@mat.ug.edu.pl (A. Zastrow).

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Some further historical remarks. The idea to generalize covering space theory is pursued (at least) since the sixties [19] to very recent propositions — see [21–23,18,13,14,12] (cf. the final remark of the introduction) for such recent work or for applications — with various authors, driven by different motivations, having made technically non-equivalent and also non-homeomorphic propositions for the constructions of generalized covering spaces. Some references before 2007 can be found in the introduction of [17]. Also various categories have been used, e.g. [4,9] and [5] for constructions in the category of uniform or metric spaces and [1–3] for constructions in the category of topological groups.

However, the majority of constructions have been performed in the topological category, and there the majority of authors used as start-point of their construction the observation that covering spaces are fibrations with discrete fibres and relaxed the discreteness-conditions for the fibres. However, a smaller group of authors [26,7,6,17,8] applied the construction of the universal path space (as it was called in [7] and as it is sometimes also used in classical covering theory [20, proof of Thm. 10.2]) also beyond the semilocal simply connected environment. However note that even amongst this group of authors different topologies have been suggested for the generalized covering spaces. In this paper we are referring to generalized covering spaces as they have been defined in [17], which is the same topology as it is used in [7] and [26], but not in [6] or [8] (observe that while the definition of [17] gives non-homeomorphic fibres [17, §1.2], the definitions of [8, §4] and [6] force homeomorphic fibres, cf. the proof of [8, Thm. 4.6], [6, bottom of p. 362] and [17, Lemma 2.1]).

In the construction that will be used in this paper, the points in a generalized universal covering space $\tilde{X}$ of a space X are defined as equivalence classes of paths of the spaces, where all paths under consideration start at the base point of the space X, and two paths give the same point of $\tilde{X}$ if they end at the same point and are relatively homotopic to each other. The basis of the topology of $\tilde{X}$ is defined by sets that are denoted by symbols $B(\alpha, U)$, where U is an open subset of X and α is a (homotopy class of) path(s) starting the base point of X and ending somewhere in U. Such a symbol represents all paths that are homotopic to the concatenation of α and of an arbitrary path entirely lying in U. This construction allows a natural projection $p : \tilde{X} \to X$ via projecting each path to its endpoint, and allows for the discussion whether this projection permits (even in the absence of evenly covered neighborhoods, that can only be expected for semilocally simply connected X) the lifting of paths as in classical covering theory. It turns out that p is always continuous (even if in general it will not be a local homeomorphism), and that each path $w : [0,1] \to X$ can be continuously lifted in a natural way: the point $w(t)$ lifts to the initial segment $w|_{[0,t]}$. However, depending on the topology of X there might be additional ways for continuously lifting paths ("skew-lifts"). Ref. [17] and the current paper take the point of view, that the above described construction of $\tilde{X}$ (which corresponds to the "universal path space" of [7]) should be only regarded as generalized covering space of X provided, it satisfies the unique path-lifting property. In [17] it was pointed out that, despite the absence of evenly covered neighborhoods, unique path-lifting guarantees that $\tilde{X}$ will be simply connected and that $\pi_1(X)$ acts by homeomorphisms as deck-transformations on $\tilde{X}$ [17, Prop. 2.14(a)]. For locally path-connected X the quotient space $\tilde{X}/\pi_1(X)$ will be homeomorphic to X again [17, Prop. 2.14(c)], for non-locally path-connected X it will be X with a topology-change that was in [8, Def. 2.17 & Thm. 5.6] called "universal Peano space" or "Peanification", and in [17, Rem. 4.17] the local "path-connectification".

By the above discussion it should be clear that (dis)proving the existence of generalized universal covering spaces is equivalent to showing that the universal path space (which can be constructed for any path-connected space) turns out to (not) having the unique path-lifting property.

Already in [10, Def. 5.2] is was observed that if the condition (that we now call "homotopically Hausdorff") is not fulfilled, even a Hausdorff space X will generate a non-Hausdorff topology for $\tilde{X}$ (hence the name), and thus even the constant path will have non-constant lifts. It is therefore a necessary condition for the existence of generalized universal covering spaces. However, by lack of corresponding examples it was long time not clear, whether it was also sufficient. The only condition which could be phrased in a similar spirit and proven to be sufficient is now called "homotopically path-Hausdorff" (cf. [16, Def. 2.10]). See [16, §6(15)] for a proof of this fact in the modern terminology and [26, Thm. 1.2] for the original proof (and note that in [26], a preprint preceding [17], the properties of being "homotopically (path-)Hausdorff" in Def. 1.1 were named differently).

The paper [16, Section 6] discusses the relations between various of these properties that turned to be relevant in classical or generalized covering theory. New and known results in this area have been gathered together in the diagram (Fig. 8). The authors of [16] also tried to investigate for each arrow in this diagram which was not an equivalence, whether the converse might also hold. For the two arrows with numbers (15) and (19) they did not decide about the reversibility. The example of this paper now shows that the implication

$$X \text{ homotopically path-Hausdorff} \overset{(15)}{\Longrightarrow} \exists \text{ generalized universal covering space of } X$$

is not reversible. We remark that it was announced that also the other of these implications, namely

$$\exists \text{ generalized universal covering space of } X \overset{(19)}{\Longrightarrow} X \text{ homotopically Hausdorff}$$

is non-reversible [27].

Definition 1. ([11]) A space X is called *homotopically Hausdorff* if for every $x_0 \in X$ and for every non-trivial $\gamma \in \pi_1(X, x_0)$ there exists a neighborhood U of x_0 such that no loop in U is homotopic (in X) to γ, rel x_0.