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A new construction of universal spaces for asymptotic dimension

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ABSTRACT

For each n, we construct a separable metric space $\mathbb{U}_n$ that is universal in the coarse category of separable metric spaces with asymptotic dimension (asdim) at most n and universal in the uniform category of separable metric spaces with uniform dimension (udim) at most n. Thus, $\mathbb{U}_n$ serves as a universal space for dimension n in both the large-scale and infinitesimal topology. More precisely, we prove:

$$\operatorname{asdim} \mathbb{U}_n = \operatorname{udim} \mathbb{U}_n = n$$

and such that for each separable metric space X,

a) if $\operatorname{asdim} X \leqslant n$, then X is coarsely equivalent to a subset of $\mathbb{U}_n$;
b) if $\operatorname{udim} X \leqslant n$, then X is uniformly homeomorphic to a subset of $\mathbb{U}_n$.

1. Introduction

For each n, we construct a space that is universal in the coarse category of separable metric spaces with asymptotic dimension at most n. Such spaces were previously constructed by Dranishnikov and Zarichnyi in [1]. Our aim is to give a more transparent construction that highlights the micro–macro analogy between small and large scales.

Our main result is the following theorem.

Theorem 1.1. *For each n, there exists a separable metric space $\mathbb{U}_n$ such that*

$$\operatorname{asdim} \mathbb{U}_n = \operatorname{udim} \mathbb{U}_n = n$$

and such that for each separable metric space X the following conditions are satisfied.

a) *If* $\operatorname{asdim} X \leqslant n$, *then* X *is* coarsely equivalent *to a subset of* $\mathbb{U}_n$.
b) *If* $\operatorname{udim} X \leqslant n$, *then* X *is* uniformly homeomorphic *to a subset of* $\mathbb{U}_n$.

The main instrument that we use is a uniform limit, which for our purposes serves the role of a large-scale analog of an inverse limit. In the first part of the paper we prove that a canonical map into a uniform limit of a suitably chosen anti-Čech approximation of a space is a coarse equivalence. In the second part we construct a sequence into which every

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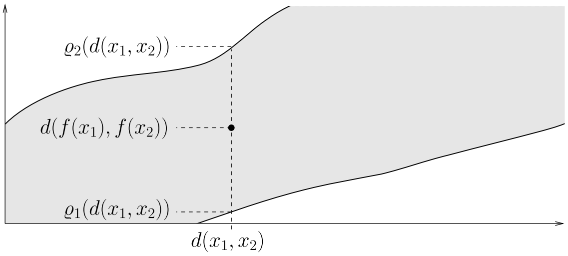

Fig. 1. The relation between ϱ_1, f and ϱ_2.

such anti-Čech approximation of an at most n-dimensional space embeds isometrically. At small scales the same arguments apply.

The space $\mathbb{U}_n$ that we construct is self-similar in small and large scales. In particular, the asymptotic cone of $\mathbb{U}_n$ is isometric to $\mathbb{U}_n$ (see Remark 4.5). We leave the following two questions open.

Question 1.2. Let $\mathbb{U}_n$ be the space constructed in the proof of Theorem 1.1.

(1) Is $\mathbb{U}_n$ an n-dimensional Polish absolute extensor in dimension n?
(2) Is $\mathbb{U}_n$ strongly universal in dimension n, i.e., is every map from an n-dimensional Polish space into $\mathbb{U}_n$ approximable by embeddings?

Characterization theorems of [4,5] state that any space with the above properties is homeomorphic to the universal n-dimensional Nöbeling space.

Let $\mathcal{U}$ be an open cover of the metric space X. Recall that the mesh of $\mathcal{U}$ is $\sup\{\operatorname{diam} U \mid U \in \mathcal{U}\}$. The multiplicity (or order) of $\mathcal{U}$ is the largest n so that there is a collection of n elements of $\mathcal{U}$ with non-empty intersection. A number $\epsilon > 0$ is said to be a Lebesgue number for $\mathcal{U}$ if every subset of X with diameter less than ϵ is contained in a member of $\mathcal{U}$.

The asymptotic Assouad–Nagata dimension, AN-asdim, is an asymptotic version of the Assouad–Nagata dimension (see, for example [2]). For a metric space X, we define $\text{AN-asdim}\, X \leqslant n$ if there is a $c > 0$ and an $r_0 > 0$ so that for each $r \geqslant r_0$ there is a cover $\mathcal{U}$ of X such that $\operatorname{mesh}(\mathcal{U}) < cr$, $\operatorname{mult}(\mathcal{U}) \leqslant n+1$, with Lebesgue number greater than r. Many results concerning the asymptotic dimension can be transferred to corresponding results concerning asymptotic Assouad–Nagata dimension, so a natural question is the following.

Question 1.3. Is there an analogous construction for asymptotic Nagata–Assouad dimension?

2. Preliminaries

The asymptotic dimension of a metric space was introduced by Gromov [3] in his study of the large-scale geometry of Cayley graphs of finitely generated groups. The definition given there is a large-scale analog of Ostrand's characterization of covering dimension. Asymptotic dimension is an invariant of coarse equivalence (see below) and Roe has shown that it can be defined not only for metric spaces, but also for so-called coarse spaces [6] (although we will content ourselves here with separable metric spaces). Extrinsic interest in asymptotic dimension was piqued by G. Yu [7], who showed that the famous Novikov higher signature conjecture holds for groups with finite asymptotic dimension. This result has subsequently been strengthened, but nevertheless, asymptotic dimension remains an intrinsically interesting invariant of the asymptotic approach to topology.

Definition. Let X be a metric space with a fixed metric d and let n be an integer. We say that

(1) the *uniform covering dimension* $\operatorname{udim} X$ of X is less than or equal to n, if for each $r > 0$ there exists an open cover of X with mesh smaller than r, positive Lebesgue number and multiplicity at most $n+1$;
(2) the *asymptotic dimension* $\operatorname{asdim} X$ of X is less than or equal to n, if for each $r < \infty$ there exists an open cover of X with Lebesgue number greater than r, finite mesh and multiplicity at most $n+1$.

Definition. Let f be a map from a metric space X into a metric space Y. Assume that $\varrho_1, \varrho_2 : (0, \infty) \to [0, \infty]$ are functions for which the following inequalities hold for each $x_1, x_2 \in X$ (see Fig. 1):

$$\varrho_1\big(d(x_1, x_2)\big) \leqslant d\big(f(x_1), f(x_2)\big) \leqslant \varrho_2\big(d(x_1, x_2)\big).$$

We say that f is