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Compactly metrizable spaces and a theorem on generalized strong Σ-spaces

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ABSTRACT

We prove that any jointly metrizable on compacta space X with a countable $k\omega$-network has a countable network (Corollary 2.3). Theorem 3.3 states that, for any continuous mapping $g : X \to Y$ of a paracompact p-space X onto a jointly metrizable on compacta space Y, there exist a metrizable space Z, a perfect mapping $f : X \to Z$, and a continuous mapping $h : Z \to Y$ such that $g = h \circ f$. It follows that a Lindelöf Σ-space is a *JCM*-space if and only if it has a countable network.

1. Introduction

This paper is a continuation of the articles [3], [4], and [5]. In terminology and notation we follow [8] and [6]. A space is *submetrizable* if its topology contains a metrizable topology.

Before 1950, one of the strong driving forces in General Topology had been, for a long time, the famous metrization problem. This problem can be formulated as follows: find a necessary and sufficient condition under which the topology of a space is generated by a metric. A very satisfactory solution of the metrization problem was given independently by J. Nagata, Ju.M. Smirnov, and R.H. Bing (see [7,10,12,8,6]).

However, there is an aspect of the metrization problem which has been overlooked until recently. Let us briefly describe it. Every space has some metrizable subspaces. Sometimes, the collection of all metrizable subspaces of a space X is rather large, even when the space X itself is not metrizable.

It is natural to consider the following general question: given a family $\mathcal{F}$ of subspaces of a space X, under what conditions there is a metric d on the set X such that d *metrizes every subspace of* X *which belongs to* $\mathcal{F}$, that is, the restriction of d to A generates the subspace topology on A, for any $A \in \mathcal{F}$? If such a metric d on X does exist, then, following [3,4], we call the space X $\mathcal{F}$*-jointly metrizable*.

A necessary condition for X to be $\mathcal{F}$-jointly metrizable is that every $A \in \mathcal{F}$ is metrizable. But this may occur due to the existence of a metric d_A on every $A \in \mathcal{F}$ which metrizes A and depends on A, while nothing guarantees that these "partial" metrics d_A can be blended into one metric d defined on the set X, the restrictions of which to members of $\mathcal{F}$ metrize all of them simultaneously.

In [3] and [4] the following special version of joint metrizability has been introduced and studied. A space X is *jointly metrizable on compacta*, or *compactly metrizable*, or X is a *JCM-space*, if there exists a metric d on X which metrizes every compact subspace F of X.

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Clearly, every submetrizable space is a *JCM*-space [3,4]. However, the class of Hausdorff *JCM*-spaces is much wider than the class of submetrizable spaces. This is witnessed by the following statement from [5], since every countably compact submetrizable space is metrizable and compact.

Proposition 1.1. *([5]) There exists a non-metrizable countably compact Tychonoff space X which is jointly metrizable on compacta.*

It was demonstrated in [3], [4], and in [5] that many theorems on metrizable spaces do not generalize to jointly metrizable on compacta spaces. In particular, there exists a separable jointly metrizable on compacta space which is not Lindelöf. On the other hand, there are also results in [3], [4], and [5] showing that some classical theorems on metrizable spaces can be extended to jointly metrizable on compacta spaces.

In this article, we extend some classical results on paracompact submetrizable spaces to the spaces which are jointly metrizable on compacta. In particular, we generalize the theorem in [2] saying that every submetrizable paracompact p-space is metrizable. A new concept of a $k\omega$-network for a space is introduced. We prove that any jointly metrizable on compacta space X with a countable $k\omega$-network has a countable network (Corollary 2.3). We also establish that if a Lindelöf Σ-space X is a *JCM*-space, then X has a countable network.

2. Some generalized networks for spaces

Below we say that a decreasing sequence $\{P_i\colon i \in \omega\}$ of subsets of a space X *converges* to a subset F of X if for every open neighborhood U of F there exists $k \in \omega$ such that $P_k \subset U$.

Suppose that $\mathcal{F}$ is a family of subsets of a space X. We will say that $\mathcal{F}$ is a *sequential k-network for X at* $x \in X$ if there exist a decreasing sequence $\eta = \{P_i\colon i \in \omega\}$ of members of $\mathcal{F}$ and a compact subspace F of X such that $x \in \cap\{P_i\colon i \in \omega, i > 0\} \subset F$, and η converges to F.

Furthermore, suppose that d is a metric on the set X. Fix also an arbitrary positive natural number n.

A function f on the family $\mathcal{F}$ will be called (d,n)*-splitting*, if for any $P \in \mathcal{F}$, $f(P)$ is some finite covering of P by subsets of P such that $\operatorname{diam}(K) < 1/n$ for each $K \in f(P)$, if such a covering of P exists; otherwise $f(P) = \emptyset$.

According to this definition, a (d,n)-splitting function on $\mathcal{F}$ always exists, and often there are many of them.

Proposition 2.1. *Suppose that X is a space, $\mathcal{F}$ is a family of its subsets, and d is a metric on X which metrizes every compact subspace of X. Suppose further that f_n is some (d,n)-splitting function on $\mathcal{F}$, for every positive $n \in \omega$, and let Y be the set of all $x \in X$ such that $\mathcal{F}$ is a sequential k-network for X at x.*

Then the family $\mathcal{S} = \bigcup\{f_n(P)\colon P \in \mathcal{F},\ n \in \omega,\ n > 0\}$ is a network for X at any $x \in Y$.

Proof. Take any $x \in Y$. We can fix a decreasing sequence $\eta = \{P_i\colon i \in \omega\}$ of members of $\mathcal{F}$ and a compact subspace F of X such that $x \in \bigcap\{P_i\colon i \in \omega,\ i > 0\} \subset F$, and η converges to F.

Claim 1. *For each positive number ϵ, there exists $n(\epsilon) \in \omega$ such that $P_{n(\epsilon)} \subset O_\epsilon(F)$.*

Assume the contrary. Then we can fix $x_n \in P_n \setminus O_\epsilon(F)$, for each $n \in \omega$. The subspace $B = F \cup \{x_n\colon n \in \omega\}$ is compact, since F is compact and η converges to F. Therefore, d metrizes B. It is also clear that some $y \in F$ is an accumulation point for the sequence $\{x_n\colon n \in \omega\}$. Hence, $d(y, \{x_n\colon n \in \omega\}) = 0$. However, it follows from $\{x_n\colon n \in \omega\} \subset (X \setminus O_\epsilon(F))$ and $y \in F$ that $d(y, \{x_n\colon n \in \omega\}) \geqslant \epsilon > 0$, a contradiction.

Take an arbitrary positive $i \in \omega$. By Claim 1, there exists $n(i) \in \omega$ such that $P_{n(i)} \subset O_{1/5i}(F)$. Clearly, we can also assume that $n(i) > i$. Since F is compact, the metric d metrizes F, and there exists a finite covering ν of F such that $\operatorname{diam}(C) < 1/5i$ for every $C \in \nu$. Clearly, $O_{1/5i}(F) = \bigcup\{O_{1/5i}(C)\colon C \in \nu\}$ and $\operatorname{diam}(O_{1/5i}(C)) < 1/i$ for any $C \in \nu$. Put $\mu = \{P_{n(i)} \cap O_{1/5i}(C)\colon C \in \nu\}$. It follows from $P_{n(i)} \subset O_{1/5i}(F)$ that $P_{n(i)} = \bigcup \mu$ and that $\operatorname{diam}(C) < 1/i$ for each $C \in \mu$. Hence, $f_i(P_{n(i)}) \neq \emptyset$, that is, $f_i(P_{n(i)})$ is a finite covering of $P_{n(i)}$ by subsets of $P_{n(i)}$ such that $\operatorname{diam}(K) < 1/i$ for each $K \in f_i(P_{n(i)})$.

Since $x \in P_{n(i)}$, we have $x \in H_i$, for some $H_i \in f_i(P_{n(i)})$. Clearly, $H_i \in \mathcal{S}$.

In this way, we have defined a sequence $\{H_i\colon i \in \omega,\ i > 0\}$ of members of $\mathcal{S}$ such that $\operatorname{diam}(H_i) < 1/i$ and $x \in H_i$. It remains to establish the next fact:

Claim 2. *Take an arbitrary open neighborhood $O(x)$ of x. Then $x \in H_k \subset O(x)$, for some positive $k \in \omega$.*

Assume the contrary. Then we can fix $x_i \in H_i \setminus O(x)$ for every positive $i \in \omega$. The subspace $\Phi = F \cup \{x_n\colon n \in \omega,\ n > 0\}$ of X is compact, since η converges to F. Therefore, d metrizes the subspace Φ. We have $d(x, x_i) < 1/i$, since $x, x_i \in H_i$ and $\operatorname{diam}(H_i) < 1/i$. Hence, $d(x, \{x_n\colon n \in \omega,\ n > 0\}) = 0$. Since d metrizes Φ, it follows that $x \in \overline{\{x_n\colon n \in \omega,\ n > 0\}}$, which is impossible, since $O(x)$ is an open neighborhood of x such that $O(x) \cap \{x_n\colon n \in \omega,\ n > 0\} = \emptyset$. This contradiction completes the proof. $\square$