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Spectral decompositions of spaces induced by spectral decompositions of acting groups

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ABSTRACT

The presence on a space of an additional algebraic structure, which is compatible with its topology, in many cases imposes strong restrictions on the properties of the space itself. In the paper we discuss connections between spectral characterizations of groups and spaces with their actions when the latter are near to their coset spaces. The compatible system of maps on a space which is induced by a compatible system of maps on the acting group is constructed. This approach allows to give a unified possibility in the study of spaces with special actions of groups with suitable spectral decompositions. Among them are Čech complete groups, their products and subgroups. In particular, it is proved that a pseudocompact coset space of a Čech-complete group is $\varkappa$-metrizable and a compactum which is a coset space of a subgroup of a product of Čech-complete groups is an openly generated compactum. We also show that an arbitrary d-open action of an $\aleph_0$-balanced group on a pseudocompact space can be replaced by a d-open action of an $\aleph_0$-bounded group.

1. Introduction and preliminaries

The notion of a projective system introduced by P.S. Alexandroff (the partial case of the notion of an inverse system) led to the appearance of inverse system's method in the investigations of topological spaces and their properties. The constructions of spaces with the prescribed properties and the investigations of complicated spaces approximating them by simple ones became their main applications in topology. A bright example of the first sort's usage is the V.V. Fedorchuk's method of developable systems and fully closed maps. Important examples of the latter usage are the S. Mardešić's theorem that any compactum is a limit space of an inverse system of compacta which dimensions don't exceed the dimension of the original compactum and its metrizable analog – the theorem of H. Freudenthal. L.S. Pontryagin gave the spectral characterization of compact topological groups – their decompositions into Lie series. His idea of transfinite inverse system's continuity allowed R. Haydon to give the spectral characterization of Dugundji compacta. The further progress of the inverse system's method in the investigations of compacta was realized in the E.V. Schepin's works [28,27,29]. He proved the spectral theorem about the isomorphism of cofinal subsystems of uncountable transfinite systems which limit spaces are homeomorphic; solved the problem of adequacy of classes of compacta to classes of maps; introduced the class of openly generated or $\varkappa$-metrizable compacta.

The inverse system's method is also successfully used in the non-compact case (E.G. Skliarenko, B.A. Pasynkov, A.V. Arhangel'skii, A.C. Chigogidze, M.G. Tkachenko, D.B. Shahmatov, V. Valov, V. Kulpa and others). An analysis of the Schepin's characterization of Dugundji compacta allowed V.V. Uspenskii [33] to define a class of (od-)d-spaces – the class

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of non-compact spaces which corresponds to the class of Dugundji compacta. The introduced notion made it possible to consider topological groups, products of spaces with a countable network and Dugundji compacta from a unified viewpoint. Every topological group is an *od*-space [33]; a coset space of an arbitrary group by a uniform subgroup is an *od*-space [17]. A coset space of an $\aleph_0$-balanced group is an *od*-space [33]; a pseudocompact G-space with a d-open action of an $\aleph_0$-balanced group is a d-space and its Stone–Čech compactification is a Dugundji compactum [17].

The cited above results of equivariant topology give confidence to suppose that the presence on a space of an additional algebraic structure, which is compatible with its topology, in many cases imposes strong restrictions on the properties of the space itself. Considering those actions of groups on spaces which define the topologies of the latter, one can expect that some topological properties of a group are transferred to a space on which it acts. The purpose of the present paper is to study connections between spectral decompositions of groups and spaces on which they act. It is shown what compatible systems of maps (in particular, equivariant) on a space correspond to the compatible systems of maps (first of all, homomorphisms) on an acting group. This approach allows to give a unified possibility in the study of spaces with special actions of groups with suitable spectral decompositions, in particular, Čech complete groups, their products and subgroups.

The main results are: the spectral decomposition of a space with a d-open continuous action of a subgroup of a Čech-complete group (Theorem 2.5); the transference of a compatible system of maps on a group which is a subgroup of a product of Čech-complete groups onto a space in case of a continuous d-open action (Corollaries 4.6, 4.13, 4.16). It is established that in case of a d-open (additionally totally bounded) action of a Čech-complete group (a subgroup of a product of Čech-complete groups) on a pseudocompact space the latter is $\varkappa$-metrizable (Corollaries 2.8, 4.8). In particular, a compact coset space of a subgroup of a product of Čech-complete groups is an openly generated compactum. In Corollaries 2.8, 4.8 it is also shown when the Stone–Čech and the maximal equivariant G-compactification of a pseudocompact space is a Dugundji or an openly generated compactum. In Corollary 5.5 it is shown how to pass from a d-open action of an $\aleph_0$-balanced group on a pseudocompact space to a d-open action of an $\aleph_0$-bounded group.

The basis of the construction how to transfer a compatible system of maps on a group onto a space is given in Theorems 1.17, 3.3, 4.5, 4.12, 4.15, Proposition 4.1 and Corollary 3.4. In Theorem 5.4 we give general condition when the action of an $\aleph_0$-balanced group can be replaced by the action of an $\aleph_0$-bounded group preserving its d-openness.

All spaces are assumed to be Tychonoff, maps are continuous, and notations, terminology and designations are from [11]. By a neighborhood we always understand an open neighborhood; $\mathrm{cl}_X A$, $\mathrm{int}_X A$ are the closure, interior of a subset A of a space X, respectively; $\mathbb{N}$ are natural numbers. The Stone–Čech compactification of X is βX. Any metric on a topological space is compatible with its topology. A metrizable space is called Polish if it is separable and has a complete metric. For the definition of $\varkappa$-metrizability see [29]. For the product $\Pi\{X_\alpha\colon \alpha \in A\}$ and $B \subset A$ the subproduct $\Pi\{X_\alpha\colon \alpha \in B\}$ is called a face. If $|B| \leqslant \tau$ then the subproduct is called a τ-face.

For covers $u = \{U_\alpha\colon \alpha \in A\}$ and $v = \{V_\beta\colon \beta \in B\}$ the notation $u \succ v$ means that u is a refinement of v. The star of a point x with respect to a cover u we denote $\mathrm{St}(x, u)$. For a family $u = \{U_\alpha\colon \alpha \in A\}$ and a subset M we put $u \wedge M = \{U_\alpha \cap M\colon \alpha \in A\}$ and $\bigcup u = \bigcup\{U_\alpha\colon \alpha \in A\}$. For a cover $\gamma = \{O_\alpha\colon \alpha \in A\}$ of a set Y and a map $f : X \to Y$ we put $f^{-1}\gamma = \{f^{-1}O_\alpha\colon \alpha \in A\}$.

1.1. *Uniform and pseudouniform structures*

The uniform structures on spaces are introduced by the families of covers [14] and are compatible with their topology. The base of any *pseudouniformity* $\mathcal{U}$ on a space X consists of open covers. Thus the topology $\tau_{\mathcal{U}}$ generated on X by $\mathcal{U}$ is weaker than the original one.

If $\mathcal{U}$ is a pseudouniformity on X then the subsets $[x]_{\mathcal{U}} = \bigcap\{\mathrm{St}(x, \upsilon) : \upsilon \in \mathcal{U}\}$ form a partition $E(\mathcal{U})$ of X. On the quotient set $X/E(\mathcal{U})$ with respect to this partition the *quotient uniformity* $\bar{\mathcal{U}}$ is defined. It is the greatest uniformity on $X/E(\mathcal{U})$ such that the quotient map $p : X \to X/E(\mathcal{U})$ is uniformly continuous (see, for example, [21]). In this case the map p is called a *uniform quotient map*. *Uniform quotient space* $X/\mathcal{U}$ is the quotient set $X/E(\mathcal{U})$ with topology induced by the quotient uniformity $\bar{\mathcal{U}}$. We denote by $X/E(\mathcal{U})$ the quotient space X with respect to the partition $E(\mathcal{U})$ and by q the natural quotient map of X onto $X/E(\mathcal{U})$. Thus the diagram

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "q"'] & & X/\mathcal{U} \\ & X/E(\mathcal{U}) \arrow[ur] & \end{tikzcd}$$

is evidently commutative.

Proposition 1.1. *If a subset A is open (closed) in the topology $\tau_{\mathcal{U}}$ induced by the pseudouniformity $\mathcal{U}$ on X then $p^{-1}p(A) = A$ and its image $p(A)$ is open (closed) in the uniform quotient space $X/\mathcal{U}$.*

If a subset A is open (closed) in $X/\mathcal{U}$ then its preimage $p^{-1}A$ is open (closed) in the topology $\tau_{\mathcal{U}}$ induced by the pseudouniformity $\mathcal{U}$ on X.

Proof. We shall give a proof for an open subset O. The proof for a closed subset may be given passing to their complement.