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On contraction-type assumptions avoiding the Hausdorff distance

Pavel V. Semenov [1]

Department of Mathematics, Moscow City Pedagogical University, 2-nd Sel'skokhozyastvennyi pr. 4, Moscow, 129226, Russia

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ABSTRACT

In this note we introduce the contraction-type assumptions for multivalued mappings and prove some fixed points theorems without using of the Hausdorff distances between subsets of a metric space.

1. Preliminaries

Typically, a fixed point theorem for multivalued mappings goes back to some fixed point theorem for single-valued mappings. Such a correlation basically deals with the substitution of a given metric, say d, on a space X by the corresponding Hausdorff "metric" $Haus_d$ on the set of all closed subsets of X. It seems, S. Nadler Jr. [4] was first who replaced the Banach contraction assumption

$$d\big(f(x), f(y)\big) \leqslant C \cdot d(x, y) < d(x, y),$$

by its multivalued analog

$$Haus_d\big(F(x), F(y)\big) \leqslant C \cdot d(x, y) < d(x, y)$$

and proved the fixed point theorem for such a mapping $F : X \to X$ with nonempty closed values $F(x) \subset X$ in a complete metric space $(X; d)$.

An analogous replacement for F. Browder type inequalities

$$Haus_d\big(F(x), F(y)\big) \leqslant \varphi\big(d(x, y)\big) < d(x, y)$$

or, for E. Racotch type inequalities

$$Haus_d\big(F(x), F(y)\big) \leqslant k\big(d(x, y)\big) \cdot d(x, y) < d(x, y)$$

for various kinds of numerical functions $\varphi : [0; \infty) \to [0; \infty)$, $k : [0; \infty) \to [0; 1)$ one can find in a lot of papers, see e.g. [1–3,5], etc.

E-mail address: pavels@orc.ru.

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Let us emphasize that one of the oldest open problems in the area is the following question, stated by S. Reich in 1974.

Question 1.1. Let $k : [0; \infty) \to [0; 1)$ and $\forall t > 0$, $\limsup_{s \to t+0} k(s) < 1$. Is it true that any k-contraction of a complete metric space has a fixed point?

The answer is affirmative for compact-valued contractions [6] and for closed-valued contractions but with the substitution $t \geqslant 0$ instead of $t > 0$ in the assumption above [3].

In this note we show that the using of the Hausdorff distance is, in general, a superfluous assumption for an existence of a sequence of successful approximations tending to a fixed point of $F : X \to X$, $x \in F(x)$. It appears, that one can directly use the *distance function*

$$d_F(x) = dist\big(x; F(x)\big), \quad x \in X$$

rather than $Haus_d$. See [7,8] for somewhat similar approaches.

Definition 1.2. For a point $x \in X$ of a metric space $(X; d)$ and for $A \subset X$ a sequence $\{y_m\}$ of points $y_m \in A$ is said to be $(x; A)$-sequence if

$$0 < d(x, y_m) \to dist(x, A) = \inf\big\{d(x, y):\ y \in A\big\}, \quad m \to \infty.$$

Note, the following are equivalent:

(1) For $x \in X$ and $A \subset X$ there exists an $(x; A)$-sequence;
(2) The point x is not an isolated point of A.

Definition 1.3. For a multivalued mapping $F : X \to X$ a sequence $\{y_m\}$ is said to be $(x; F)$-sequence if it is $(x; F(x))$-sequence.

So, for a numerical function $\varphi : (0; +\infty) \to (0; +\infty)$ a multivalued mapping $F : X \to X$ of a metric space $(X; d)$ is said to be **sequential φ-contraction** if for every $x \in X$ there exists an $(x; F)$-sequence $\{y_m\}$ such that $d_F(y_m) < \varphi(d(x; y_m))$, $m \in \mathbb{N}$.

2. Statements

Theorem 2.1. *For any nondecreasing function $\varphi : (0; +\infty) \to (0; +\infty)$ and for any closed-valued sequential φ-contraction $F : X \to X$ of a metric space $(X; d)$ one of the following two statements is true*:

(*a*) *F has a fixed point*;
(*b*) *for each sufficiently large $t > 0$ there are points $x_n \in X$ such that for all $n \in \mathbb{N}$ the following are true*

$$x_{n+1} \in F(x_n); \qquad 0 < d(x_n; x_{n+1}) < \varphi^n(t); \qquad 0 < d_F(x_{n+1}) < \varphi^{n+1}(t).$$

Proof. Let us check that the negotiation of (a) implies (b). So, suppose that a closed-valued sequential φ-contraction $F : X \to X$ admits no fixed points, i.e. $d_F(x) > 0$ for all $x \in X$. A construction of a desired sequence $\{x_n\}$ for (b) follows to the natural inductive scheme.

(0) Let x_0 be an arbitrary point of X, $x_0 \notin F(x_0) \Leftrightarrow d_F(x_0) > 0$.
(1) In accordance with the notion of sequential φ-contraction pick any $(x_0; F)$-sequence, say $\{y_m\}$, such that $d_F(y_m) < \varphi(d(x_0; y_m))$, $m \in \mathbb{N}$. In particular, for $x_1 = y_1 \in F(x_0)$ we see that

$$0 < d(x_0; x_1), \qquad 0 < d_F(x_1) < \varphi\big(d(x_0; x_1)\big)$$

and for an arbitrary picked $t > d(x_0; x_1)$ we obtain:
(i_1) $0 < d(x_0; x_1) < t = \varphi^0(t)$;
(ii_1) $0 < d_F(x_1) < \varphi(d(x_0; x_1)) \leqslant \varphi(t) = \varphi^1(t)$.
$(n+1)$ Let for every $1 \leqslant k \leqslant n$ the point x_k $F(x_{k-1})$ was chosen with the properties that
(i_k) $0 < d(x_{k-1}; x_k) < \varphi^{k-1}(t)$;
(ii_k) $0 < d_F(x_k) < \varphi^k(t)$.
In accordance with the notion of sequential φ-contraction pick any $(x_n; F)$-sequence, say $\{z_m\}$, and for $\varepsilon = \varphi^n(t) - d_F(x_n) > 0$ choose an appropriate $x_{n+1} = z_{m(n)} \in F(x_n)$ such that
(i_{n+1}) $0 < d(x_n; x_{n+1}) < d_F(x_n) + \varepsilon = \varphi^n(t)$;
(ii_{n+1}) $0 < d_F(x_{n+1}) < \varphi(d(x_n; x_{n+1})) \leqslant \varphi(\varphi^n(t)) = \varphi^{n+1}(t)$. □