Topology and its Applications 160 (2013) 960–964

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The weights of closed subrings of a locally compact ring

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ARTICLE INFO

Article history:
Received 25 November 2012
Received in revised form 8 March 2013
Accepted 8 March 2013

MSC:
16W80

Keywords:
Weight
Locally compact ring
Almost connected ring
Nilring of bounded degree
Suitable set

ABSTRACT

We prove that if R is a locally compact nondiscrete ring, then for every cardinal number $\mathfrak{m}$, $\aleph_0 \leqslant \mathfrak{m} \leqslant w(R)$ there exists a closed subring S such that $w(S) = \mathfrak{m}$.

1. Introduction

Our paper is inspired by recent results of S. Hernández, K.H. Hofmann, and S.A. Morris (see [6]). We extend the main results of the paper [6] to locally compact associative rings. If R is a compact nilpotent ring of uncountable weight, then for every cardinal number $\mathfrak{m}$, $\aleph_0 \leqslant \mathfrak{m} < w(R)$ there exists a closed ideal I such that $w(I) = \mathfrak{m}$. Examples 3.1 and 3.2 show that this theorem is not true for all compact rings. It is known that every infinite associative ring contains a commutative infinite subring [9]. We give an independent proof of this result for compact rings. We will use freely the classical result that every totally disconnected compact ring is profinite (see, e.g., [7,11]).

2. Notation and conventions

All rings are assumed associative not necessarily with identity. Topological rings and groups are assumed to be Hausdorff. The Jacobson radical of a ring R is denoted by $J(R)$. If R is a ring, then its annihilator $Ann(R)$ is the ideal $\{x \in R\colon xR = Rx = 0\}$. The symbol R_0 stands for the connected component of zero of a topological ring R. If X is a subset of a ring R, then $\langle X \rangle$ means the subring of R generated by X.

3. Results

Recall [4, p. 596 and p. 601] that a subset X of a topological group G is called *suitable* if

(i) X topologically generates G.
(ii) The identity element $1 \notin X$ and X is discrete and closed in $G \setminus \{1\}$.

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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.03.003

Furthermore, the *generating rank* of a compact group G is the minimal cardinal number $\aleph$ for which there is a suitable set X of G such that $\operatorname{card} X = \aleph$. It is denoted by $s(G)$.

We will need the following results [4, Theorem 12.11 and Proposition 12.28]:

Theorem 3.1. *Every compact group G has a suitable set.*

Theorem 3.2. *If G is an infinite totally disconnected compact group, then*

$$w(G) = \max\{\aleph_0, s(G)\}.$$

Theorem 3.3. *If R is a compact infinite ring, then for every infinite cardinal $\mathfrak{m} < w(R)$ there exists a closed subring S with $w(S) = \mathfrak{m}$.*

Proof. We can consider without loss of generality that $w(R) > \aleph_0$.

Let R_0 be the connected component of R. By Kaplansky's Theorem [7, Theorem 8], $RR_0 = R_0R = 0$. If $w(R_0) \geqslant \mathfrak{m}$, then we can find a closed subgroup of R_0 of weight $\mathfrak{m}$, which automatically will be a subring. Therefore we can assume that $w(R_0) < \mathfrak{m}$. Then $w(R) = w(R/R_0)$. It suffices to find a closed subring S of R/R_0 of weight $\mathfrak{m}$.

Therefore we can consider that R is totally disconnected. Let X be a suitable set of the additive group of R. Then $\operatorname{card}(X) = w(R) > \mathfrak{m}$. Let Y be a subset of X of cardinality $\mathfrak{m}$. The set Z of all nonzero finite products of elements of Y has cardinality $\mathfrak{m}$. The closed subring S generated by Y, has the subset Z as a set of topological generators of its additive subgroup and is suitable. It follows that $w(S) = \mathfrak{m}$. □

Theorem 3.4. *Every infinite compact nilring R contains an infinite ideal I for which $I^2 = 0$.*

Proof. We can consider without loss of generality that R is totally disconnected.

Let I be a maximal ideal in the set of ideals with trivial multiplication. If I is infinite, the proof is finished. If I is finite, then there exists an open ideal V such that $V \cap I = 0$. By [12], R is a nilring of bounded degree. Then by the Lewitzki–Herstein Theorem [10, p. 180], V contains a nonzero nilpotent ideal N. If M is the ideal of R generated by N, then by Andrunachievich's Lemma, $M^3 \subset N$, hence M is nilpotent. We can assume that $M^2 = 0$. Then $(I + M)^2 = 0$ and $I \subset_{\neq} I + M$, a contradiction. □

Theorem 3.5. *Let R be a compact nilpotent ring and $w(R) > \aleph_0$. Then for every cardinal number $\mathfrak{m}$, $\aleph_0 \leqslant \mathfrak{m} \leqslant w(R)$ there exists a closed ideal I such that $w(I) = \mathfrak{m}$.*

Proof. Induction on the index n of nilpotence. For $n = 2$, the result follows from [6]. Assume that the assertion was proved for each nilpotent ring of index of nilpotence $\leqslant n$ and let R be a compact nilpotent ring of index of nilpotence $n + 1$. Then $cl(R^n) \subset Ann(R)$. If $w(cl(R^n)) = \mathfrak{m}$, then every subgroup of $cl(R^n)$ is an ideal of R and we can take a closed subgroup of $cl(R^n)$ of weight $\mathfrak{m}$. If $w(cl(R^n)) < \mathfrak{m}$, then $w(R/cl(R^n)) = w(R)$. By the assumption we can find a closed ideal I' of weight $\mathfrak{m}$. Denote by $q : R \to R/cl(R^n)$ the canonical homomorphism. Then $w(I) = \mathfrak{m}$, where $I = q^{-1}(I')$. □

Lemma 3.6. *Let $\mathfrak{M}$ be the variety of associative rings defined by the identities $2x = 0 = x^2$. Then $\mathfrak{M}$ has no identities of the form $x_{n+1} f(x_1, \dots, x_n) = 0$, where $f(x_1, \dots, x_n) = 0$ is not an identity on $\mathfrak{M}$.*

Proof. Let $X = \{x_i\colon i \in \mathbb{N}\}$ and I be the ideal of the ring $R = \mathbb{F}_2[X]$ of polynomials generated by X. Let K be the ideal of R generated by the set $\{x_i^2\colon i \in \mathbb{N}\}$. Then I/K is the free ring in the variety $\mathfrak{M}$ with $y_i = x_i + K$, $i \in \mathbb{N}$, as free generators. Since $y_{n+1} f(y_1, \dots, y_n) \neq 0$, the proof is finished. □

Example 3.1. A compact nilring having no infinite metrizable nilpotent ideals.

Let X be an arbitrary set of cardinality $> 2^{\aleph_0}$ and let $\mathfrak{X}$ be the family of all finite nonempty subsets of X. Let $F(X)$ be the free ring over X in the variety $\mathfrak{M}$ of associative rings given by identities $2x = 0 = x^2$.

Denote for each $Y \in \mathfrak{X}$ by $F(Y)$ the subring of $F(X)$ generated by Y. Obviously, every ring in $\mathfrak{M}$ is commutative and $F(Y)$ is the free ring in $\mathfrak{M}$ over Y. Consider the family $\{F(Y)\}_{Y \in \mathfrak{X}}$ as an inverse system of (finite) rings as follows: If $Y_1 \subset Y_2$ and $Y_1, Y_2 \in \mathfrak{X}$, then $p_{Y_2,Y_1} : F(Y_2) \to F(Y_1)$ maps y in y for all $y \in Y_1$, and all y to 0 for all $y \in Y_2 \setminus Y_1$.

Let $L = \varprojlim F(Y)$ be the inverse limit. We claim that L has the needed properties. Denote by p_Y, where $Y \in \mathfrak{X}$ the projection of L onto $F(Y)$.

It suffices to show that the ideal Ll is nonmetrizable for every $0 \neq l \in L$. Since $l \neq 0$, there exists $Y \in \mathfrak{X}$ such that $p_Y(l) \neq 0$. That means that $p_Y(l) = f(x_1, \dots, x_n)$, where $Y = \{x_1, \dots, x_n\}$. Let $y, z \in X \setminus Y$, $y \neq z$. Set $U = Y \cup \{y\}$, $V = Y \cup \{z\}$, and $W = Y \cup \{y, z\}$. Then $p_U(l) = f(x_1, \dots, x_n) + yg(x_1, \dots, x_n)$, $p_V(l) = f(x_1, \dots, x_n) + zh(x_1, \dots, x_n)$, and $p_W(l) = f(x_1, \dots, x_n) + ys(x_1, \dots, x_n) + zt(x_1, \dots, x_n) + yzq(x_1, \dots, x_n)$, where $f(x_1, \dots, x_n), g(x_1, \dots, x_n), h(x_1, \dots, x_n), s(x_1, \dots, x_n), t(x_1, \dots, x_n), q(x_1, \dots, x_n) \in F(Y)$.