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Absolutely strongly star-Menger spaces

Yan-Kui Song [1]

Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing 210023, PR China

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ABSTRACT

A space X is *absolutely strongly star-Menger* if for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X and each dense subset D of X, there exists a sequence $(F_n\colon n \in N)$ of finite subsets of D such that $\{St(F_n, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is an open cover of X. In this paper, we investigate the relationships between absolutely strongly star-Menger spaces and related spaces, and also study topological properties of absolutely strongly star-Menger spaces.

1. Introduction

By a space we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Fleischman [9] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F, \mathcal{U}) = X$, where $St(F, \mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon U \cap F \neq \emptyset\}$. He proved that every countably compact space is starcompact. van Douwen et al. in [6] showed that every T_2 starcompact space is countably compact, but this does not hold for T_1-spaces (see [15, Example 2.5]). Matveev [14] defined a space X to be *absolutely countably compact* (= acc) if for each open cover $\mathcal{U}$ of X and each dense subset D of X, there exists a finite subset F of D such that $St(F, \mathcal{U}) = X$. It is clear that every T_2 absolutely countably compact space is countably compact.

Matveev [13] defined a space X to be *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $St(F, \mathcal{U}) = X$. Bonanzinga [1,2] defined a space X to be *absolutely star-Lindelöf* if for each open cover $\mathcal{U}$ of X and each dense subset D of X, there exists a countable subset F of D such that $St(F, \mathcal{U}) = X$. It is clear that every absolutely star-Lindelöf space is star-Lindelöf.

In [6], a starcompact space is called strongly starcompact and a star-Lindelöf space is called strongly star-Lindelöf.

Kočinac [11,12] defined a space X to be *strongly star-Menger* if for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X, there exists a sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of X such that $\{St(F_n, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is an open cover of X.

Caserta, Di Maio and Kočinac [5] gave the selective version of the notion of acc spaces and defined the class of the following spaces.

E-mail address: songyankui@njnu.edu.cn.

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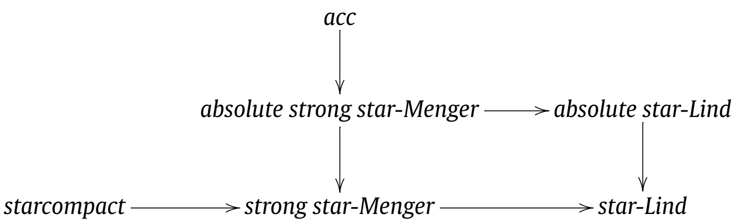
Definition. ([5]) A space X is said to be *absolutely strongly star-Menger* if for each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X and each dense subset D of X, there exists a sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of D such that $\{St(F_n, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is an open cover of X.

From the above definitions, we have the following digram.



The purpose of this paper is to investigate the relationships between absolutely strongly star-Menger spaces and related spaces, and study topological properties of absolutely strongly star-Menger spaces.

Throughout this paper, the *extent* $e(X)$ of a space X is the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of real numbers. For each ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma\colon \alpha \leqslant \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma\colon \alpha < \gamma \leqslant \beta\}$, $(\alpha, \beta) = \{\gamma\colon \alpha < \gamma < \beta\}$ and $[\alpha, \beta] = \{\gamma\colon \alpha \leqslant \gamma \leqslant \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [8].

2. Absolutely strongly star-Menger spaces and related spaces

In this section, we give some examples showing relationships between absolutely strongly star-Menger spaces and related spaces.

Example 2.1. There exists a Tychonoff absolutely strongly star-Menger space X which is not acc.

Proof. Let

$$X = \big([0, \omega_1] \times [0, \omega]\big) \setminus \big\{\langle \omega_1, \omega\rangle\big\}$$

be the Tychonoff plank [8]. Then X is not countably compact, since $\{\langle \omega_1, n\rangle\colon n \in \omega\}$ is a countable discrete closed subset of X. Hence X is not acc.

Next we show that X is absolutely strongly star-Menger. To this end, let $\{\mathcal{U}_n\colon n \in \mathbb{N}\}$ be a sequence of open covers of X. Let S be the set of all isolated points of $[0, \omega_1]$ and $D = S \times [0, \omega)$. Then D is a dense subset of X and every dense subset of X includes D. Thus it is sufficient to show that there exists a sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of D such that $\{St(F_n, \mathcal{U}_n)\colon n \in \mathbb{N}\}$ is an open cover of X. Since $[0, \omega_1)$ is acc by Theorem 1.8 in [14], then $[0, \omega_1) \times [0, \omega]$ is acc by Theorem 2.3 in [14]. For each $n \in \mathbb{N}$, there exists a finite subset F'_n of D such that

$$[0, \omega_1) \times [0, \omega] \subseteq St\big(F'_n, \mathcal{U}_n\big).$$

On the other hand, for each $n \in \mathbb{N}$, there exists some $U_n \in \mathcal{U}_n$ such that $\langle \omega_1, n-1\rangle \in U_n$. We can pick $d_n \in D$ such that $d_n \in U_n$. Then $\langle \omega_1, n-1\rangle \in St(d_n, \mathcal{U}_n)$. For each $n \in \mathbb{N}$, let $F_n = F'_n \cup \{d_{n-1}\}$. Then the sequence $(F_n\colon n \in \mathbb{N})$ of finite subsets of D witnesses for $(\mathcal{U}_n\colon n \in \mathbb{N})$ that X is absolutely strongly star-Menger. □

Remark 2.2. The space X of Example 2.1 also shows that there exists a Tychonoff absolutely strongly star-Menger space X which is not countably compact (hence not starcompact).

Example 2.3. There exists a countably compact (hence starcompact) Tychonoff space X which is not absolutely strongly star-Menger.

Proof. Let $X = [0, \omega_1) \times [0, \omega_1]$ be the product of $[0, \omega_1)$ and $[0, \omega_1]$. Clearly, X is countably compact, hence starcompact.

We show that X is not absolutely strongly star-Menger. For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha) \times (\alpha, \omega_1] \quad \text{and} \quad D = [0, \omega_1) \times [0, \omega_1).$$

For each $n \in \mathbb{N}$, let

$$\mathcal{U}_n = \{U_\alpha\colon \alpha < \omega_1\} \cup \{D\}.$$