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Tunnel number degeneration under the connected sum of prime knots

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ABSTRACT

We study 2-string free tangle decompositions of knots with tunnel number two. As an application, we construct infinitely many counter-examples to a conjecture in the literature stating that the tunnel number of the connected sum of prime knots doesn't degenerate by more than one: $t(K_1 \# K_2) \geqslant t(K_1) + t(K_2) - 1$, for K_1 and K_2 prime knots.

1. Introduction

Given a knot K in S^3, an *unknotting tunnel system* for K is a collection of arcs $t_1, t_2, \dots, t_n$, properly embedded in the exterior of K, with the complement of a regular neighborhood of $K \cup t_1 \cup \dots \cup t_n$ being a handlebody.[2] The minimum cardinality of an unknotting tunnel system for a knot K is a knot invariant, referred to as the *tunnel number* of K and is denoted by $t(K)$.

A natural question of study on knot invariants is their behavior under the connected sum of knots. In the particular case of the tunnel number, it is known, by Norwood [20], that tunnel number one knots are prime. This result is now consequence of more general work. For instance, Scharlemann and Schultens prove in [25] that the tunnel number of the connected sum of knots is bigger than or equal to the number of summands:

$$t(K_1 \# \cdots \# K_n) \geqslant n,$$

where $K_1 \# \cdots \# K_n$ represents the connected sum of the knots $K_1, \dots, K_n$. Also, in [3] Gordon and Reid prove that tunnel number one knots are, in fact, n-string prime[3] for any positive integer n.

On the tunnel number behavior under connected sum, it is a consequence from the definition of connected sum of knots that for two knots K_1 and K_2 in S^3 we have:

$$t(K_1 \# K_2) \leqslant t(K_1) + t(K_2) + 1.$$

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[2] Note that every knot has an unknotting tunnel system obtained from the knot exterior triangulation.

[3] A knot is n-string prime if it has no n-string essential tangle decomposition. For definitions on tangle decompositions see Definitions 2 and 3 in Section 2 (Preliminaries).

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For some time the only examples known had an additive behavior:

$$t(K_1\#K_2) = t(K_1) + t(K_2).$$

However, in the early nineties, Morimoto [12] constructed connected sum examples of prime knots K_1 with 2-bridge knots K_2 whose tunnel number degenerates by one[4]:

$$t(K_1\#K_2) = t(K_1) + t(K_2) - 1.$$

Shortly afterwards, Moriah and Rubinstein in [11], and also independently Morimoto, Sakuma and Yokota in [16], gave examples of knots with supper-additive behavior:

$$t(K_1\#K_2) = t(K_1) + t(K_2) + 1.$$

Furthermore, about the same time, Kobayashi in [6] constructed examples of knots that degenerate arbitrarily under connected sum: for any positive integer n, there are knots K_1 and K_2 where

$$t(K_1\#K_2) \leqslant t(K_1) + t(K_2) - n.$$

However, Kobayashi's examples to show arbitrarily hight degeneration of the tunnel number under connected sum are composite knots.

In this paper we study further the tunnel number degeneration under connected sum of prime knots. For this study we use the work of Morimoto in [15] that relates n-string free tangle decompositions of knots and high tunnel number degeneracy under the connected sum of prime knots. Within this setting, we study 2-string free tangle decompositions of knots with tunnel number two and we obtain Theorem 1, and its Corollary 1.1, for which statement we need the following definition.

Definition 1. Let s be a properly embedded arc in a ball B. Suppose the knot obtained by capping off s along ∂B has tunnel number at most one. We say that s is *μ-primitive* if there is an unknotted arc t properly embedded in B, disjoint from s, such that the tangle $(B, s \cup t)$ is free.

Remark 1. Note that a string s is μ-primitive if, and only if, the knot obtained by capping off s along ∂B is the unknot or a μ-primitive tunnel number one knot.[5]

Theorem 1. *Let K be a tunnel number two knot with a 2-string free tangle decomposition. Then both strings of some tangle are μ-primitive.*[6]

Corollary 1.1. *Let K be a knot with a 2-string free tangle decomposition where no tangle has both strings being μ-primitive. Then* $t(K) = 3$.

The only examples of prime knots whose tunnel number degenerates under connected sum are the ones given by Morimoto, and in this case the tunnel number only degenerates by one. Also, Kobayashi and Rieck in [7], and also Morimoto in [14], proved that the tunnel number of the connected sum of m-small[7] knots doesn't degenerate. With this and other results in perspective, Moriah conjectured in [10] that the tunnel number of the connected sum of prime knots doesn't degenerate by more than one: $t(K_1\#K_2) \geqslant t(K_1) + t(K_2) - 1$, for K_1 and K_2 prime knots.

In this paper, we construct infinitely many counter-examples to this conjecture as in Theorem 2 and its Corollary 2.1.

Theorem 2. *There are infinitely many tunnel number three prime knots K_1 such that, for any 3-bridge knot K_2, $t(K_1\#K_2) \leqslant 3$.*

Corollary 2.1. *There are infinitely many prime knots K_1 and K_2 where*

$$t(K_1\#K_2) = t(K_1) + t(K_2) - 2.$$

In [26], Scharlemann and Schultens introduced the concept of degeneration ratio for the connected sum of two prime knots, K_1 and K_2:

4 In [15], without mentioning it, Morimoto gives also the first examples of knots that when connected sum with themselves the tunnel number degenerates (by one): all tunnel number two 3-bridge knots with a 2-string free tangle decomposition (as the knot 10_{149} from Rolfsen's list in [23]).

5 For a definition of *μ-primitive* knot see Definition 5.13 of the survey paper [10] by Moriah.

6 The correspondent result to Theorem 1 for links exists and is also proved by the author in [19].

7 A knot is said m-small if there is no incompressible surface with meridional boundary components in its complement.