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Note on the cohomology of finite cyclic coverings

Yasuhiro Hara [a], Daisuke Kishimoto [b,*,1]

[a] Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043, Japan
[b] Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan

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ABSTRACT

We introduce the height of a normal cyclic p-fold covering and show a cohomological relation between the base and the total spaces of the covering in terms of the height. We also interpret the height in terms of the category weight.

1. Statement of results

The purpose of this note is to show a cohomological property of a normal cyclic p-fold covering with respect to a certain cup-length type invariant of the covering. Let p be a prime and let $E \to B$ be a normal cyclic p-fold covering where B is path-connected. Suppose $p = 2$. In [8], Kozlov defined the *height* of the covering $\mathrm{h}(E)$ as the maximum n such that $w_1(E)^n \neq 0$, where $w_1(E)$ is the first Stiefel–Whitney class of the covering. By a chain level consideration, he proved

$$H^{\mathrm{h}(E)}(E; \mathbb{Z}/2) \neq 0.$$

This also follows immediately from the Gysin sequence of the double covering $E \to B$. We would like to generalize this result to any prime p. Let p be an arbitrary prime. Let C_p be a cyclic group of order p and let $\rho : B \to BC_p$ be the classifying map of the covering $E \to B$. The *height* of the covering can be generalized as

$$\mathrm{h}(E) = \max\bigl\{n \bigm| \rho^* : H^n(BC_p; \mathbb{Z}/p) \to H^n(B; \mathbb{Z}/p) \text{ is non-trivial}\bigr\}.$$

We remark here that the height of a normal cyclic p-fold covering is closely related with the ideal-valued cohomological index theory of Fadell and Husseini [5] and hence the Borsuk–Ulam theorem. We will interpret the height in terms of the category weight introduced by Fadell and Husseini [6] and studied further by Rudyak [12] and Strom [13]. The most difficult point in generalizing the result of Kozlov is the non-existence of the Gysin sequence for the covering $E \to B$ when p is odd. However, we define the corresponding spectral sequence by which we prove:

* Corresponding author.
E-mail addresses: hara@math.sci.osaka-u.ac.jp (Y. Hara), kishi@math.kyoto-u.ac.jp (D. Kishimoto).

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Theorem 1.1. *Let $E \to B$ be a normal cyclic p-fold covering, where B is path-connected. Then*

$$H^{\mathrm{h}(E)}(E;\mathbb{Z}/p) \neq 0.$$

As an immediate corollary, we have:

Corollary 1.2. *Let $E \to B$ be a normal cyclic p-fold covering, where B is path-connected. If $\mathrm{h}(E) \geqslant n$ and $H^n(E;\mathbb{Z}/p) = 0$, it holds that $\mathrm{h}(E) \geqslant n+1$.*

In Section 2, we construct a spectral sequence for a normal cyclic p-fold covering which calculates the mod p cohomology of the total space from the base space whose differential is shown to be given as a certain higher Massey product of Kraines [9]. Using this spectral sequence, we prove Theorem 1.1. In section 3, we interpret the height of a normal cyclic p-fold covering in terms of the category weight introduced by Fadell and Husseini [6] and elaborated by [12] and [13].

2. Proof of Theorem 1.1

Throughout this section, let p be an odd prime and the coefficient of cohomology is $\mathbb{Z}/p$.

2.1. Spectral sequence

Let $E \to B$ be a normal p-fold covering where B is path-connected. In this subsection, we introduce a spectral sequence which calculates the mod p cohomology of E from B. Analogous spectral sequences were considered in [4] and [11]. We first set notation. Let $\rho : B \to BC_p$ be the classifying map of the covering $E \to B$. Recall that the mod p cohomology of BC_p is given as

$$H^*(BC_p) = \Lambda(u) \otimes \mathbb{Z}/p[v], \qquad \beta u = v, \qquad |u| = 1,$$

where β is the Bockstein operation. We denote the cohomology classes $\rho^*(u)$ and $\rho^*(v)$ of B by $\bar{u}$ and $\bar{v}$, respectively. Let $R[C_p]$ denote the group ring of C_p over a ring R. Note that the singular chain complex $S_*(E)$ is a free $\mathbb{Z}[C_p]$-module. We regard $\mathbb{Z}/p[C_p]$ and $\mathbb{Z}/p$ as $\mathbb{Z}[C_p]$-modules by the modulo p reduction and the trivial C_p-action, respectively. Then there are natural isomorphisms

$$H^*\big(\mathrm{Hom}_{\mathbb{Z}[C_p]}\big(S_*(E), \mathbb{Z}/p[C_p]\big)\big) \cong H^*(E) \quad \text{and} \quad H^*\big(\mathrm{Hom}_{\mathbb{Z}[C_p]}\big(S_*(E), \mathbb{Z}/p\big)\big) \cong H^*(B). \tag{2.1}$$

We now fix a generator g of C_p and put $\tau = 1 - g \in \mathbb{Z}/p[C_p]$. Observe that $\mathbb{Z}/p[C_p] = \mathbb{Z}/p[\tau]/(\tau^p)$. Consider the filtration

$$0 \subset \tau^{p-1}\mathbb{Z}/p[C_p] \subset \tau^{p-2}\mathbb{Z}/p[C_p] \subset \cdots \subset \tau\mathbb{Z}/p[C_p] \subset \mathbb{Z}/p[C_p].$$

Then there is a spectral sequence (E_r, d_r) associated with the induced filtration of the cochain complex $\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p[C_p])$. By (2.1), we have

$$E_1^{s,t} \cong \begin{cases} H^t(B), & 0 \leqslant s \leqslant p-1, \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad H^*(E) \tag{2.2}$$

and the degree of the differential d_r is $(-r, 1)$, where the total degree of $E_r^{s,t}$ is t. Let us identify the differential of this spectral sequence. To this end, we calculate the induced coboundary map $\bar{\delta}$ of the associated graded cochain complex

$$\bigoplus_{i=0}^{p-1} \mathrm{Hom}_{\mathbb{Z}[C_p]}\big(S_*(E), \tau^i\mathbb{Z}/p[C_p]/\tau^{i-1}\mathbb{Z}/p[C_p]\big) \cong \bigoplus_{i=0}^{p-1} \tau^i \,\mathrm{Hom}_{\mathbb{Z}}\big(S_*(B), \mathbb{Z}/p\big).$$

In the special case of the universal bundle $EC_p \to BC_p$, we may put

$$\bar{\delta}(1) = \tau u_1 + \cdots + \tau^{p-1} u_{p-1}, \qquad u_i \in \mathrm{Hom}_{\mathbb{Z}}\big(BC_p, \mathbb{Z}/p\big)$$

for $1 \in \mathrm{Hom}_{\mathbb{Z}}(BC_p, \mathbb{Z}/p)$. Consider the map $E \xrightarrow{\tilde{\rho}\times\pi} EC_p \times B$, where $\tilde{\rho}$ is a lift of ρ and π is the projection. Then we see that

$$\bar{\delta}x = \delta x + \tau\rho^*(u_1)x + \cdots + \tau^{p-1}\rho^*(u_{p-1})x \tag{2.3}$$

for any $x \in \mathrm{Hom}_{\mathbb{Z}}(S_*(B), \mathbb{Z}/p)$ in general. If $[u_1] = 0$, $1 \in E^{1,0}$ becomes a permanent cycle in the spectral sequence (2.2) for the universal bundle $EC_p \to BC_p$, which contradicts to the contractibility of EC_p. Then by normalizing u if necessary, we may assume

$$[u_1] = u. \tag{2.4}$$