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The main theorem of discrete Morse theory for Morse matchings with finitely many rays

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ABSTRACT

The main theorem of discrete Morse theory states that a finite, regular CW complex X equipped with a discrete Morse function is homotopy equivalent to a CW complex that has one d-cell for each critical cell in X of index d. We prove, using the terminology of discrete Morse matchings, a version of this theorem that works for infinite complexes, provided the Morse matching induces finitely many equivalence classes of rays in the Hasse diagram. We work in the class of h-regular posets, introduced by Minian, which is strictly larger than the class of face posets of regular CW complexes. A homological version of the theorem for cellular posets is also given.

1. Introduction

Discrete Morse theory, developed by Robin Forman (see [1], or [2] for a non-technical introduction), is a tool for investigating homotopy type and homology groups of finite CW complexes. It is a discrete analogue of the classical, smooth Morse theory, introduced by Marston Morse in the twenties [3], and later intensively developed and often applied in an outstanding way. The smooth Morse theory gives relations between critical points of a smooth map defined on a manifold and the topology of the manifold. In particular, given a compact manifold M and a smooth map $f: M \to \mathbb{R}$ whose critical points are all non-degenerate (such a map is called a Morse map), we may conclude that M is homotopy equivalent to a CW complex X whose d-cells are in bijective correspondence with critical points of f of index d, for all $0 \leqslant d \leqslant \dim M$. Moreover, if f has no critical values in some interval $[a,b]$, then the manifolds $M(a) = f^{-1}((-\infty,a])$ and $M(b) = f^{-1}((-\infty,b])$ are diffeomorphic, and $M(a)$ is a deformation retract of $M(b)$. A classical reference for the smooth Morse theory is Milnor's book [4].

In the discrete setting Forman calls a function f from the set of cells of a regular, finite CW complex X to the set of real numbers $\mathbb{R}$ a discrete Morse function, if for each cell σ the sets

$$u_f(\sigma) = \{\tau > \sigma\colon \dim(\tau) = \dim(\sigma) + 1,\ f(\tau) \leqslant f(\sigma)\}$$

and

$$d_f(\sigma) = \{\mu < \sigma\colon \dim(\mu) = \dim(\sigma) - 1,\ f(\mu) \geqslant f(\sigma)\}$$

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have cardinality at most one. A cell σ is called critical with respect to f if both $u_f(\sigma)$ and $d_f(\sigma)$ are empty. It turns out [1], and this result is called the main theorem of discrete Morse theory, that the regular CW complex X is homotopy equivalent to a CW complex that has exactly one n-dimensional cell for each critical cell in X of dimension n. Moreover, for $a \in \mathbb{R}$ one can define the level subcomplex

$$X(a) = \bigcup_{f(\sigma)\leqslant a} \bigcup_{\tau\leqslant\sigma} \tau.$$

If $f^{-1}([a,b])$ does not contain any critical cells for some $a < b \in \mathbb{R}$, then the level subcomplex $X(b)$ collapses (in the sense of simple homotopy theory [5]) to $X(a)$. Note that these theorems are discrete analogues of the smooth results. (For a comparison of the smooth and the discrete Morse theory see [6].)

Discrete Morse theory has found many applications, mainly in topological combinatorics and in applied mathematics. It has also been extended in different directions and presented from different viewpoints. For some extensions of the theory developed by Forman himself, consider [7,8]. An approach to equivariant version of the theory was taken by Ragnar Freij [9]. Gabriel E. Minian [10] extends the theory to a class of posets strictly larger than the class of face posets of regular CW complexes. Some algebraic versions of the theory, such as presented in [9,11,12], are also available.

However, most of these extensions deal with finite CW complexes. The problem of extending the theory to infinite complexes was stated by Forman in [2] and tackled with by several authors in [13–16]. The present paper deals with the same problem, with a slightly different approach. (It is perhaps worth noting that the discrete Morse theory was also applied to infinite complexes in [17], though in a vastly different spirit than in the present paper; we shall not discuss these investigations.)

To compare our results with those of [13–15] we use an observation often attributed to Manoj K. Chari [18], that a discrete Morse function f on a finite CW complex X induces a matching M on the Hasse diagram $\mathcal{H}(X)$ of the face poset of X. The matching is constructed using the fact that for a cell σ of X only one of the sets $u_f(\sigma), d_f(\sigma)$ may be nonempty; its only element is matched with σ. This matching is acyclic, which means that the digraph $\mathcal{H}_M(X)$ obtained from $\mathcal{H}(X)$ by inverting the arrows which belong to the matching does not contain a directed cycle. Those acyclic matchings are in fact a different but equivalent description of discrete vector fields, studied by Forman [1], and they carry all the information on critical cells and the sets $d_f(\sigma), u_f(\sigma)$. (The links between discrete Morse theory and graph theory by means of Morse matchings have been studied by several authors. For a recent study see [19].)

In the case when the CW complex X is finite, given an acyclic matching M on X one may construct a discrete Morse function on X whose associated matching is M. In [13] the authors extend these results to the case of locally finite complexes. They consider discrete Morse functions that are proper, which means that $f^{-1}([a,b])$ is finite for all $a < b \in \mathbb{R}$.

For proper discrete Morse functions the proofs contained in [1] may be applied to show that if $f^{-1}([a,b])$ does not contain any critical cells for $a < b \in \mathbb{R}$, then the level subcomplexes $X(a), X(b)$ are homotopy equivalent. This allows one to study the topology of X at different levels of f, but does not give a description or an explicit construction of a complex built solely of the critical cells. In [14,15] the discrete Morse inequalities were given for proper discrete Morse functions on 1-dimensional and 2-dimensional locally finite complexes, which may be considered a step in this direction.

Call an infinite, directed simple path in $\mathcal{H}_M(X)$ a decreasing ray. Two decreasing rays are said to be equivalent if they coincide from some point. The paper [13] characterizes the matchings coming from proper discrete Morse functions as those that are acyclic and do not contain some specific configurations of infinite paths in the graph $\mathcal{H}_M(X)$, though this is only done in case of matchings with a finite number of critical cells and equivalence classes of decreasing rays.

In the present article we take a different approach and concentrate on the discrete Morse matchings, forgetting about discrete Morse functions. We prove that given an acyclic matching M on an infinite (not necessarily locally finite), regular CW complex X such that $\mathcal{H}_M(X)$ contains finitely many equivalence classes of decreasing rays, one may construct a CW complex X_M that has exactly one cell for each critical cell of X and one cell for each equivalence class of decreasing rays in X_M. This yields a version of the discrete Morse inequalities as a corollary. Since we do not assume local finiteness or finite dimensionality of the CW complex under investigation, our results extend those of [14,15].

In fact we prove a bit more than it is stated above. In [10] G. Minian introduced the classes of admissible posets and homologically admissible posets, which are both strictly larger than the class of face posets of regular CW complexes. Our results hold for the class of admissible posets and moreover we prove a homological version of the theorem for homologically admissible posets.

The paper is organized as follows. In Section 2 we give some notation used throughout the paper and define basic combinatorial concepts used in discrete Morse theory. Section 3 defines rayless discrete Morse matchings and proves some of their basic properties. Next, in Section 4 we show that given a discrete Morse matching M that induces finitely many equivalence classes of decreasing rays, one can create a new matching that is rayless and apart from the critical cells coming from M has one new critical cell for each equivalence class of decreasing rays in M. In Section 5 we derive from the work of Freij [9], which was one of the main inspirations for writing this article, a homological version of the main theorem of discrete Morse theory for Morse matchings with finitely many equivalence classes of decreasing rays. Section 6 gives a homotopical version of the theorem. Finally, in Section 7 we give some concluding remarks and ideas for further research.