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The Big Bush machine

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α-Space
σ-Relatively discrete dense subset
Dense metrizable subspace
σ-Disjoint base
σ-Point-finite base
Quasi-development
G_δ-diagonal
Base of countable order
β-Space
p-Space
Σ-space
Strong completeness properties
Countable regular co-compactness
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Weakly α-favorable
Almost base-compact
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ABSTRACT

In this paper we study an example-machine *Bush*(S, T) where S and T are disjoint dense subsets of $\mathbb{R}$. We find some topological properties that *Bush*(S, T) always has, others that it never has, and still others that *Bush*(S, T) might or might not have, depending upon the choice of the disjoint dense sets S and T. For example, we show that every *Bush*(S, T) has a point-countable base, is hereditarily paracompact, is a non-Archimedean space, is monotonically ultra-paracompact, is almost base-compact, weakly α-favorable and a Baire space, and is an α-space in the sense of Hodel. We show that *Bush*(S, T) never has a σ-relatively discrete dense subset (and therefore cannot have a dense metrizable subspace), is never Lindelöf, and never has a σ-disjoint base, a σ-point-finite base, a quasi-development, a G_δ-diagonal, or a base of countable order. We show that *Bush*(S, T) cannot be a β-space in the sense of Hodel and cannot be a p-space in the sense of Arhangelskii or a Σ-space in the sense of Nagami. We show that *Bush*$(\mathbb{P}, \mathbb{Q})$ is not homeomorphic to *Bush*$(\mathbb{Q}, \mathbb{P})$. Finally, we show that a careful choice of the sets S and T can determine whether the space *Bush*(S, T) has strong completeness properties such as countable regular co-compactness, countable base compactness, countable subcompactness, and ω-Čech-completeness, and we use those results to find disjoint dense subsets S and T of $\mathbb{R}$, each with cardinality 2^ω, such that *Bush*(S, T) is not homeomorphic to *Bush*(T, S). We close with a family of questions for further study.

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1. Introduction

A topological space called the Big Bush has been an important example in the study of generalized metric base conditions (e.g., point-countable bases, σ-disjoint bases, σ-point-finite bases, and quasi-developments) and also in the study of linearly ordered topological spaces. The original Big Bush consisted of all strings of irrational numbers indexed by any countable ordinal and ending with a rational number, with the set being ordered lexicographically [2].

It is now clear that the Big Bush described above is just one of a large family of spaces called $Bush(S,T)$ where S and T are disjoint dense sets of real numbers[1] and that $Bush(S,T)$ is really a two-parameter example machine that produces linearly ordered spaces of varying degrees of complexity. There are some very strong topological properties that every $Bush(S,T)$ has, and other properties that no $Bush(S,T)$ has. Even more interesting are the properties of $Bush(S,T)$ that can be fine-tuned by careful choice of the sets S and T. The goal of this paper is to introduce the Big Bush example machine and to study the way that the descriptive properties of S, T, and $S\cup T$ control the topological properties of $Bush(S,T)$. In the process, we obtain examples that elucidate the fine structure of several strong completeness properties introduced by Choquet, de Groot, and Oxtoby.

In this paper we restrict attention to the case where S and T are disjoint non-empty subsets of the real line and we define the space $Bush(S,T)$ as follows: for each $\alpha<\omega_1$ we let

$$X(\alpha):=\bigl\{f:[0,\alpha]\to S\cup T\colon\ \beta<\alpha\Rightarrow f(\beta)\in S \text{ and } f(\alpha)\in T\bigr\},$$

and then $Bush(S,T):=\bigcup\{X(\alpha)\colon\ \alpha<\omega_1\}$. (In this notation, the original Big Bush was $Bush(\mathbb{P},\mathbb{Q})$, where $\mathbb{P}$ and $\mathbb{Q}$ are the sets of irrational and rational numbers, respectively.) Note that for each $f\in Bush(S,T)$ there is exactly one ordinal $\alpha(f)$ such that $f\in X(\alpha(f))$ and we define $\mathrm{lv}(f)=\alpha(f)$.

We linearly order the set $Bush(S,T)$ using the lexicographic order. In other words, if $f\neq g$ belong to $Bush(S,T)$ then the set $\{\alpha\colon\ f(\alpha)\neq g(\alpha)\}$ is not empty (because $S\cap T=\emptyset$). Consequently there is a first ordinal $\delta=\delta(f,g)$ such that $f(\delta)\neq g(\delta)$ and we define $f\prec g$ provided $f(\delta)<g(\delta)$ in the usual ordering of $\mathbb{R}$. This linear order gives an open interval topology on $Bush(S,T)$ in the usual way. In Section 2 we will show that basic neighborhoods of $f\in Bush(S,T)$ have a particularly simple form: for $\epsilon>0$, and $\alpha=\mathrm{lv}(f)$, let

$$B(f,\epsilon):=\bigl\{g\in Bush(S,T)\colon\ \alpha\leqslant \mathrm{lv}(g),\ g(\beta)=f(\beta) \text{ for all } \beta<\alpha,\ \text{and } \bigl|g(\alpha)-f(\alpha)\bigr|<\epsilon\bigr\}.$$

In Section 2, we show that each $B(f,\epsilon)$ is a convex open set and that the collection $\{B(f,\frac{1}{n})\colon\ n\geqslant 1\}$ is a neighborhood base at f in the open-interval topology of the linear ordering $\prec$.

Our paper is organized as follows. In Section 2 we prove a sequence of technical lemmas about $Bush(S,T)$. In Section 3 we show that whenever S and T are disjoint dense subsets of $\mathbb{R}$, the space $Bush(S,T)$ is monotonically normal, has a point-countable base, is hereditarily paracompact, is the union of ω_1-many metrizable subspaces, is a non-Archimedean space, is monotonically ultra-paracompact, is almost base-compact, pseudo-complete, α-favorable, and a Baire space, and is also an α-space in the sense of Hodel. In Section 4, we study properties that $Bush(S,T)$ cannot have (provided S and T are disjoint dense subsets of $\mathbb{R}$). We show that $Bush(S,T)$ is not Lindelöf, cannot have a σ-relatively-discrete dense subset and therefore cannot have a dense metrizable subspace, cannot have a σ-disjoint or a σ-point-finite base, cannot be quasi-developable, is not weakly perfect, and cannot be (for example) a Σ-space, a p-space, an M-space, or a β-space in the sense of Hodel. In Section 5 we explore topological properties that $Bush(S,T)$ might or might not have, depending on the descriptive structure of S, T, and $S\cup T$. This allows us to show, as a start, that the two Big Bushes $Bush(\mathbb{P},\mathbb{Q})$ and $Bush(\mathbb{Q},\mathbb{P})$ are not homeomorphic. Then we turn to the role of strong Baire-Category completeness properties in $Bush(S,T)$. We prove, for example, that $Bush(S,T)$ is countably base-compact if and only if there is a dense G_δ-subset $D\subseteq\mathbb{R}$ with $T\subseteq D\subseteq S\cup T$. That result allows us to find disjoint dense subsets S and T of $\mathbb{R}$, each with cardinality 2^ω, such that $Bush(S,T)$ is not homeomorphic to $Bush(T,S)$. We give necessary conditions for $Bush(S,T)$ to be countably subcompact, ω-Čech-complete, and strongly Choquet complete, namely that there is a dense G_δ-subset $E\subseteq\mathbb{R}$ with $E\subseteq S\cup T$. In Section 6 we list a family of open questions.

Throughout this paper, $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$ will denote the sets of real, irrational, and rational numbers with the usual ordering, and $\mathbb{Z}$ will denote the set of all integers (positive, negative, and zero). We will use the symbol $<$ for the ordering of $[0,\omega_1)$ as well as for the ordering of $\mathbb{R}$, and context will make it clear which is meant in a given situation. We reserve the symbol $\prec$ for the ordering of $Bush(S,T)$. For $f,h\in Bush(S,T)$ we will use the symbol (f,h) to denote $\{g\in Bush(S,T)\colon\ f\prec g\prec h\}$ and if $s,t\in\mathbb{R}$ we will use (s,t) for the usual open interval of real numbers. Context will make it clear whether a given interval is in $Bush(S,T)$ or in $\mathbb{R}$.

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[1] It is possible to use other kinds of disjoint subsets of a different linearly ordered space in the $Bush(S,T)$ construction, obtaining spaces with quite different properties.