Topology and its Applications 159 (2012) 2728–2735

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Lefschetz periodic point free self-maps of compact manifolds

Grzegorz Graff [a,*,1], Agnieszka Kaczkowska [a], Piotr Nowak-Przygodzki [a,1], Justyna Signerska [a,b,2]

[a] Faculty of Applied Physics and Mathematics, Gdansk University of Technology, Narutowicza 11/12, 80-233 Gdansk, Poland
[b] Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland

ARTICLE INFO

Article history:
Received 18 January 2012
Received in revised form 16 March 2012
Accepted 19 March 2012

MSC:
37C25
55M20

Keywords:
Lefschetz number
Periodic points
Rational exterior space
Cohomology ring

ABSTRACT

Let f be a self-map of a compact connected manifold M. We characterize Lefschetz periodic point free continuous self-maps of M for several classes of manifolds and generalize the results of Guirao and Llibre [J.L.G. Guirao, J. Llibre, On the Lefschetz periodic point free continuous self-maps on connected compact manifolds, Topology Appl. 158 (16) (2011) 2165–2169].

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

An important but difficult problem in dynamical systems is to identify the periodic point free maps on a given compact manifold. The necessary (but not sufficient) condition for a map to be periodic point free is that the Lefschetz numbers of all its iterates vanish. This motivates the following definition: a map f is called *Lefschetz periodic point free* iff $L(f^m) = 0$ for $m = 1, 2, 3, \ldots$. In [7] a characterization of Lefschetz periodic point free maps on $\mathbb{S}^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{S}^p \times \mathbb{S}^q$ was given. The aim of this paper is to give a more detailed description of the Lefschetz periodic point free maps on the spaces considered in [7] (Section 2), as well as to generalize these results for the large class of manifolds called rational exterior spaces (Section 3).

Our approach in this part of the paper is based on the use of additional information, hidden in the structure of the cohomology ring, which allows one to determine the sequence $\{L(f^m)\}_{m=1}^{\infty}$ (cf. [4]). This method enables us to show that some conditions found in [7] are superfluous, because they are always satisfied. Moreover, it makes it possible to find the explicit formula for $L(f^m)$ which is conceptually simpler than the use of the Lefschetz zeta function applied in [7]. We also correct some incorrect statements in [7] concerning $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

In Section 4 we formulate the necessary and sufficient conditions for a map to be Lefschetz periodic point free in the language of eigenvalues, based only on the definition of Lefschetz number. These conditions enable us to identify some spaces that do not admit Lefschetz periodic point free maps.

* Corresponding author.
E-mail addresses: graff@mif.pg.gda.pl (G. Graff), akaczkowska@mif.pg.gda.pl (A. Kaczkowska), piotrnp@wp.pl (P. Nowak-Przygodzki), j.signerska@impan.pl (J. Signerska).

1 The research supported by MSHE grant No. N N201 373236.
2 The research supported by the National Science Centre grant UMO-2011/01/N/ST1/02698.

0166-8641/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.topol.2012.03.011

In the final part of the paper (Section 5) we illustrate the way in which the purely algebraical condition of being "Lefschetz periodic point free" provides some important information about the structure of periodic points of the map.

2. Lefschetz periodic point free maps of $\mathbb{S}^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{S}^p \times \mathbb{S}^q$: cohomological ring approach

2.1. Lefschetz numbers of iterates

For a compact connected manifold M of dimension n we will consider $H^i(M;\mathbb{Q})$, where $i = 0, 1, \dots, n$, the cohomology groups with coefficients in $\mathbb{Q}$, which are then finite dimensional linear spaces over $\mathbb{Q}$. For a self-map f of M we denote by f^{*i} the linear map induced by f on $H^i(M;\mathbb{Q})$ and by f^* the self-map $\bigoplus_{i=0}^n f^{*i}$ of $\bigoplus_{i=0}^n H^i(M;\mathbb{Q})$. The Lefschetz number $L(f^m)$ of f^m is then equal to

$$L(f^m) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(f^m)^{*i}, \tag{2.1}$$

where $\operatorname{tr}(f^m)^{*i}$ is the trace of the matrix representing $(f^m)^{*i} : H^i(M;\mathbb{Q}) \to H^i(M;\mathbb{Q})$. Notice that if A is a matrix of f^{*i}, then A^m is a matrix of $(f^m)^{*i}$, representing the homomorphism induced on $H^i(M;\mathbb{Q})$ by f^m, the m-th iteration of f (cf. [2,9]). Consequently, when $\operatorname{tr} f^{*i} = \sum_{j=1}^k \lambda_j$, then $\operatorname{tr}(f^m)^{*i} = \sum_{j=1}^k \lambda_j^m$, where the sum is taken over all eigenvalues λ_j in the spectrum of A, counted with multiplicities.

Often the Lefschetz number of f is defined via homology groups, but for our purposes it is more convenient to give the equivalent definition (2.1).

We will make use of the structure of the cohomology ring $\bigoplus_{i=0} H^i(X;\mathbb{Q})$ to obtain additional information concerning the Lefschetz numbers of iterates of f. Let us recall two basic properties of cup product that we will apply later in the paper:

- f^* is a homomorphism of $\bigoplus_{i=0}^n H^i(M;\mathbb{Q})$, i.e.
$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta); \tag{2.2}$$
- the cup product is anticommutative, i.e. if $\alpha \in H^k(M;\mathbb{Q})$ and $\beta \in H^l(M;\mathbb{Q})$, then
$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha. \tag{2.3}$$

2.2. Lefschetz periodic point free maps on $\mathbb{C}P^n$

We consider the complex projective space, denoted $\mathbb{C}P^n$. The cohomology ring over $\mathbb{Q}$ of $\mathbb{C}P^n$ is isomorphic to the quotient polynomial ring $\mathbb{Q}[\alpha]/(\alpha^{n+1})$, where α is the generator of $H^2(\mathbb{C}P^n;\mathbb{Q}) = \mathbb{Q}$ (cf. e.g. [8]). As a consequence, $H^*(\mathbb{C}P^n;\mathbb{Q}) = \bigoplus_{i=0}^n H^{2i}(\mathbb{C}P^n,\mathbb{Q})$ and each $H^{2i}(\mathbb{C}P^n,\mathbb{Q})$ is generated by

$$\alpha^i = \underbrace{\alpha \smile \alpha \smile \cdots \smile \alpha}_{i} \quad \text{for } i \geqslant 0$$

where we adopt the convention that $\alpha^0 = 1$. Now, we are in a position to calculate the sequence of Lefschetz numbers of iterates (cf. similar calculation for $L(f)$ in [9]). As $(f^m)^{*2}$ has the eigenvalue a^m, where $a \in \mathbb{Z}$ is such that $f^{*2}(\alpha) = a\alpha$, we get by the formulas (2.1) and (2.2):

$$L(f^m) = 1 + \sum_{i=1}^{n} (a^m)^i = \begin{cases} \frac{1-(a^m)^{n+1}}{1-a^m} & \text{if } a^m \neq 1, \\ n+1 & \text{if } a^m = 1. \end{cases} \tag{2.4}$$

It is obvious that if $a \notin \{-1, 1\}$ then $L(f^m) \neq 0$ for $m = 1, 2, 3, \dots$. On the other hand, if $a \in \{-1, 1\}$ then either for $m = 1$ or for $m = 2$ we get $a^m = 1$ and thus by the second part of formula (2.4) $L(f^m) = n + 1 \neq 0$. As a result we obtain the following:

Proposition 2.1. *There are no Lefschetz periodic point free maps on $\mathbb{C}P^n$.*

2.3. Lefschetz periodic point free maps on $\mathbb{H}P^n$

Let $\mathbb{H}P^n$ denote the n-dimensional quaternionic projective space. The structure of the cohomology ring over $\mathbb{Q}$ is similar to $\mathbb{C}P^n$; however now the generator is four-dimensional (see e.g. [8]):

$$H^*(\mathbb{H}P^n;\mathbb{Q}) = \bigoplus_{i=0}^{n} H^{4i}(\mathbb{H}P^n,\mathbb{Q}) \simeq \mathbb{Q}[\alpha]/(\alpha^{n+1}),$$

where $\alpha \in H^4(\mathbb{H}P^n;\mathbb{Q}) \simeq \mathbb{Q}$.

Download English Version:

https://daneshyari.com/en/article/4659253

Download Persian Version:

https://daneshyari.com/article/4659253

Daneshyari.com