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On paratopological groups ☆

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ABSTRACT

In this paper, we firstly construct a Hausdorff non-submetrizable paratopological group G in which every point is a G_δ-set, which gives a negative answer to Arhangel'skiĭ and Tkachenko's question [A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Sci., 2008]. We also prove that each first-countable Abelian paratopological group is submetrizable. Moreover, we discuss developable paratopological groups and construct a non-metrizable, separable, Moore paratopological group. Further, we prove that a regular, countable, locally k_ω-paratopological group is a discrete topological group or contains a closed copy of S_ω. Finally, we discuss some properties on non-H-closed paratopological groups, and show that Sorgenfrey line is not H-closed, which gives a negative answer to Arhangel'skiĭ and Tkachenko's question [A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press and World Sci., 2008]. Some questions are posed.

1. Introduction

A *semitopological group* G is a group G with a topology such that the product map of $G \times G$ into G is separately continuous. A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ into G is jointly continuous. If G is a paratopological group and the inverse operation of G is continuous, then G is called a *topological group*. However, there exists a paratopological group which is not a topological group; Sorgenfrey line [16, Example 1.2.2] is such an example. Paratopological groups were discussed and many results have been obtained [6–8,14,19,21–23].

Proposition 1.1. *([28]) For a group with topology (G, τ) the following conditions are equivalent:*

(1) *G is a paratopological group.*
(2) *The following Pontrjagin conditions for basis $\mathscr{B} = \mathscr{B}_\tau$ of the neutral element e of G are satisfied.*

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(a) $(\forall U, V \in \mathscr{B})\ (\exists W \in \mathscr{B})$: $W \subset U \cap V$;
(b) $(\forall U \in \mathscr{B})\ (\exists V \in \mathscr{B})$: $V^2 \subset U$;
(c) $(\forall U \in \mathscr{B})\ (\forall x \in U)\ (\exists V \in \mathscr{B})$: $Vx \subset U$;
(d) $(\forall U \in \mathscr{B})\ (\forall x \in G)\ (\exists V \in \mathscr{B})$: $xVx^{-1} \subset U$.

The paratopological group G is Hausdorff if and only if
(e) $\bigcap\{UU^{-1}\colon U \in \mathscr{B}\} = \{e\}$.

The paratopological group G is a topological group if and only if
(f) $(\forall U \in \mathscr{B})(\exists V \in \mathscr{B})$: $V^{-1} \subset U$.

In this paper, we mainly discuss the following questions.

Question 1.2. ([8, Open problem 3.3.1]) Suppose that G is a Hausdorff (regular) paratopological group in which every point is a G_δ-set. Is G submetrizable?

Question 1.3. ([8, Open problem 5.7.2]) Let G be a regular first-countable ω-narrow paratopological group. Is G submetrizable?

Question 1.4. ([5, Problem 20]) Is every regular first-countable (Abelian) paratopological group submetrizable?

Question 1.5. ([5, Problem 22]) Is it true that every regular first-countable (Abelian) paratopological group G has a zero-set diagonal[1]?

Question 1.6. ([5, Problem 21]) Is every regular first-countable (Abelian) paratopological group Dieudonné complete?

Question 1.7. ([8, Open problem 3.4.3]) Let G be a regular ω-narrow first-countable paratopological group. Does there exist a continuous isomorphism of G onto a regular (Hausdorff) second-countable paratopological group?

Question 1.8. ([22]) Is a regular symmetrizable paratopological group metrizable?

Question 1.9. ([8, Open problem 5.7.5]) In every paratopological group, which is Moore space, metrizable?

Question 1.10. ([8, Open problem 3.6.5]) Must the Sorgenfrey line S be closed in every Hausdorff paratopological group containing it as a paratopological subgroup?

We shall give negative answers to Questions 1.2, 1.8, 1.9, and 1.10, and give a partial answer to Question 1.3. Moreover, we shall also give affirmative answers to Questions 1.4, 1.5, 1.6 and 1.7 when the group G is Abelian.

2. Preliminaries

Definition 2.1. Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$; (b) the family $\mathscr{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathscr{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathscr{P}_x$.

The family $\mathscr{P}$ is called a *weak base* for X [1] if, for every $A \subset X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathscr{P}_x$ such that $P \subset A$. The space X is *weakly first-countable* if $\mathscr{P}_x$ is countable for each $x \in X$.

Definition 2.2.

(1) A space X is called an S_ω-space if X is obtained by identifying all the limit points from a topological sum of countably many convergent sequences;
(2) A space X is called an S_2-space (*Arens' space*) if $X = \{\infty\} \cup \{x_n\colon n \in \mathbb{N}\} \cup \{x_n(m)\colon m, n \in \mathbb{N}\}$ and the topology is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_n(m)\colon m > k,$ for some $k \in \mathbb{N}\}$; a basic neighborhood of ∞ is $\{\infty\} \cup (\bigcup\{V_n\colon n > k$ for some $k \in \mathbb{N}\})$, where V_n is a neighborhood of x_n.

[1] We say that a space X has a *zero-set diagonal* if the diagonal in $X \times X$ is a zero-set of some continuous real-valued function on $X \times X$.